

ADELIC CARTIER DIVISORS WITH BASE CONDITIONS AND THE BONNESEN–DISKANT–TYPE INEQUALITIES

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ABSTRACT. In this paper, we introduce positivity notions for pairs of adelic \mathbb{R} -Cartier divisors and base conditions, and study fundamental properties of the arithmetic volumes associated to such pairs. As a main result, we show that the Gâteaux derivatives of the arithmetic volume function at big pairs along the directions of adelic \mathbb{R} -Cartier divisors are given by suitable arithmetic positive intersection numbers.

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1. INTRODUCTION

Let X be a normal projective variety that is geometrically irreducible over a number field K , and let $\text{Rat}(X)$ be the field of rational functions on X . We freely use the notion and basic properties of the adelic \mathbb{R} -Cartier divisors, and refer to [25] for details (see also Notation and terminology 4). To an adelic \mathbb{R} -Cartier divisor \overline{D}

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on X , we assign a finite set of all the *strictly small* sections of \overline{D} ,

$$\widehat{\Gamma}^{\text{ss}}(\overline{D}) := \left\{ \phi \in \text{Rat}(X)^\times : \overline{D} + (\widehat{\phi}) > 0 \right\} \cup \{0\},$$

and define the arithmetic volume of \overline{D} as

$$\widehat{\text{vol}}(\overline{D}) := \limsup_{\substack{m \in \mathbb{N}, \\ m \rightarrow +\infty}} \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

The arithmetic Siu inequality of Yuan [28] is essentially equivalent to the fact that the Gâteaux derivatives of the arithmetic volume function at big adelic \mathbb{R} -Cartier divisors are given by the arithmetic positive intersection numbers (see [9, 18]). It also infers the equidistribution theorem of algebraic points with small heights, and has fruitful applications to the arithmetic dynamics.

The purpose of this paper is to introduce the notion of pairs of adelic \mathbb{R} -Cartier divisors and base conditions, and to study their positivity properties. We show that the above-mentioned result on the differentiability of the arithmetic volume function can be naturally generalized to the arithmetic volume function associated to such pairs.

Let $\mathfrak{V}(\text{Rat}(X))$ be the set of all the *normalized* (non-trivial) discrete valuations of $\text{Rat}(X)$. A *base condition* is defined as a finite formal sum

$$\mathcal{V} = \sum_{\nu \in \mathfrak{V}(\text{Rat}(X))} \nu(\mathcal{V})[\nu]$$

with coefficients $\nu(\mathcal{V})$ in \mathbb{R} . We say that \mathcal{V} is *divisorial* if, for every $\nu \in \mathfrak{V}(\text{Rat}(X))$ with $\nu(\mathcal{V}) \neq 0$, the residue field of ν has the maximal transcendence degree over K (see Definitions 2.3 and 2.4).

What we study in this paper is a pair $(\overline{D}; \mathcal{V})$ consisting of an adelic \mathbb{R} -Cartier divisor \overline{D} on X and a base condition \mathcal{V} . We denote by $\widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ the \mathbb{R} -vector space of all such pairs, and set

$$\widehat{\text{DDiv}}_{\mathbb{R}, \mathbb{R}}(X) := \left\{ (\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X) : \mathcal{V} \text{ is divisorial} \right\}.$$

As in the case of adelic \mathbb{R} -Cartier divisors, we assign to such a pair $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ a finite set of the strictly small sections of \overline{D} vanishing along \mathcal{V}_+ ,

$$\widehat{\Gamma}^{\text{ss}}(\overline{D}; \mathcal{V}) := \left\{ \phi \in \text{Rat}(X)^\times : \overline{D} + (\widehat{\phi}) > 0, \nu_X(D + (\phi)) \geq \nu(\mathcal{V}) \right\} \cup \{0\}$$

(see sections 2.1 and 2.2 for detail) and define the arithmetic volume of $(\overline{D}; \mathcal{V})$ as

$$\widehat{\text{vol}}(\overline{D}; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{N}, \\ m \rightarrow +\infty}} \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

A merit of our formulation of the pairs of adelic \mathbb{R} -Cartier divisors and base conditions is that they behave well with respect to birational morphisms.

A pair $(\overline{D}; \mathcal{V})$ is said to be *big* if there exists a *weakly ample* adelic \mathbb{R} -Cartier divisor \overline{A} on X (see Notation and terminology 5 for definition of weak ampleness) such that $\overline{D} - \overline{A}$ is strictly effective and $\nu_X(D - A) \geq \nu(\mathcal{V})$ for every $\nu \in \mathfrak{V}(\text{Rat}(X))$. Our main theorem is then stated as follows.

Theorem A (Theorem 4.2). *Let X be a normal projective variety over a number field, let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ be a pair, and let \overline{D}' be an adelic \mathbb{R} -Cartier divisor on X . If $(\overline{D}; \mathcal{V})$ is big, then the Gâteaux derivative of the arithmetic volume function at $(\overline{D}; \mathcal{V})$ along \overline{D}' is given by*

$$\lim_{r \rightarrow 0} \frac{\widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V})}{r} = (\dim X + 1) \cdot \langle (\overline{D}; \mathcal{V})^{\dim X} \rangle \cdot \overline{D}'.$$

The right-hand side of the theorem is the arithmetic positive intersection number defined for pairs (see section 3.2 for detail). An *approximation* of a big pair $(\overline{D}; \mathcal{V})$ is a couple $(\mu : X' \rightarrow X, \overline{M})$ consisting of a modification $\mu : X' \rightarrow X$ and a nef and big adelic \mathbb{R} -Cartier divisor \overline{M} on X' such that $(\mu^*\overline{D} - \overline{M}; \mathcal{V}^\mu)$ is pseudo-effective (see (2.23) and Definition 2.8). We denote by $\widehat{\Theta}(\overline{D}; \mathcal{V})$ the set of all the approximations of $(\overline{D}; \mathcal{V})$. For a nef and big adelic \mathbb{R} -Cartier divisor \overline{N} , we define

$$\langle (\overline{D}; \mathcal{V})^{\dim X} \rangle \cdot \overline{N} := \sup_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}; \mathcal{V})} \widehat{\deg} \left(\overline{M}^{\dim X} \cdot \mu^* \overline{N} \right),$$

which we can extend by linearity and continuity to

$$\langle (\overline{D}; \mathcal{V})^{\dim X} \rangle \cdot : \widehat{\text{Div}}_{\mathbb{R}}(X) \rightarrow \mathbb{R}$$

(see Definition 3.4).

It is known that, if X is a smooth curve and $(\overline{D}; \mathcal{V})$ is big, then the ordered set

$$\Upsilon(\overline{D}; \mathcal{V}) := \{ \overline{P} : \overline{P} \text{ is nef and } (\overline{D} - \overline{P}; \mathcal{V}) \geq 0 \}$$

admits a unique maximal element $\overline{P}(\overline{D}; \mathcal{V})$ (see [25]). In this case, the arithmetic positive intersection numbers are given by

$$\langle (\overline{D}; \mathcal{V}) \rangle \cdot \overline{D}' = \widehat{\deg} \left(\overline{P}(\overline{D}; \mathcal{V}) \cdot \overline{D}' \right)$$

for every adelic \mathbb{R} -Cartier divisor \overline{D}' . Detail will appear elsewhere.

In the context of convex geometry, T. Bonnesen gave a systematic proof to the classical isoperimetric inequality in dimension two by showing a stronger inequality called Bonnesen's inequality (see [2]). The method was generalized to the case of arbitrary dimensions by V. I. Diskant (see [26, 11]). Analogous inequalities in the context of algebraic geometry were established by Boucksom–Favre–Jonsson (see [5, 10]), and those in the context of Arakelov geometry are in [18]. These inequalities are important in studying the properties of the volume functions and the Zariski decompositions of divisors. As a corollary of Theorem A, we can generalize the Bonnesen–Diskant–type inequalities ([18, Theorem 7.1 and Corollary 7.3]) to the case of pairs.

Theorem B (Theorem 4.9). *Let X be a normal projective variety over a number field, and let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ be big pairs. We set*

$$s_i := \langle (\overline{D}_1; \mathcal{V}_1)^i \cdot (\overline{D}_2; \mathcal{V}_2)^{(\dim X + 1 - i)} \rangle$$

for $i = 0, \dots, \dim X + 1$,

$$r = r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) := \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (\mu^* \overline{D}_1 - t\overline{M}; \mathcal{V}_1^\mu) \geq 0 \},$$

and

$$R = R((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) := \frac{1}{r((\overline{D}_2; \mathcal{V}_2), (\overline{D}_1; \mathcal{V}_1))}.$$

One then has

(1) (an arithmetic Diskant inequality)

$$0 \leq \left(s_{\dim X}^{\frac{1}{\dim X}} - r s_0^{\frac{1}{\dim X}} \right)^{\dim X+1} \leq s_{\dim X}^{1+\frac{1}{\dim X}} - s_{\dim X+1} \cdot s_0^{\frac{1}{\dim X}},$$

(2)

$$\begin{aligned} \frac{s_{\dim X}^{\frac{1}{\dim X}} - \left(s_{\dim X}^{1+\frac{1}{\dim X}} - s_{\dim X+1} \cdot s_0^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}{s_0^{\frac{1}{\dim X}}} &\leq r \\ &\leq \frac{s_{\dim X+1}}{s_{\dim X}} \leq \dots \leq \frac{s_1}{s_0} \\ &\leq R \leq \frac{s_{\dim X+1}^{\frac{1}{\dim X}}}{s_1^{\frac{1}{\dim X}} - \left(s_1^{1+\frac{1}{\dim X}} - s_0 \cdot s_{\dim X+1}^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}, \end{aligned}$$

and

(3) (an arithmetic Bonnesen inequality)

$$\left(\frac{s_0}{2} (R - r) \right)^2 \leq s_1^2 - s_0 s_2$$

if X has dimension one.

In particular, if the big pairs $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{DDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ satisfy

$$\widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2)^{\frac{1}{\dim X+1}} = \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1)^{\frac{1}{\dim X+1}} + \widehat{\text{vol}}(\overline{D}_2; \mathcal{V}_2)^{\frac{1}{\dim X+1}},$$

then $s_{\dim X}^{\dim X+1} = s_{\dim X+1}^{\dim X} \cdot s_0$, $s_1^{\dim X+1} = s_0^{\dim X} \cdot s_{\dim X}$, and

$$\left(\frac{s_{\dim X}}{s_0} \right)^{\frac{1}{\dim X}} = r = \frac{s_{\dim X+1}}{s_{\dim X}} = \dots = \frac{s_1}{s_0} = R = \left(\frac{s_{\dim X+1}}{s_1} \right)^{\frac{1}{\dim X}}.$$

The structure of this paper is as follows. After defining of the notation and terminology we use in this paper in section 1.1, we introduce, in section 2, the notion of pairs of adelic \mathbb{R} -Cartier divisors and base conditions and their positivity properties. Here we would like to treat the positivity and the ν -positivity simultaneously, because the arguments are almost parallel. The latter will be used elsewhere. The main purpose of section 2 is Theorem 2.21 asserting the openness of the big cone of pairs.

Sections 2.3 and 2.4 are devoted to study fundamental properties of the arithmetically ample adelic \mathbb{R} -Cartier divisors. We give definitions of arithmetic volumes and arithmetic base loci associated to pairs in section 2.5 and in section 2.4, respectively.

In section 3, we show some preliminary results to show the main theorems. In particular, the arithmetic positive intersection numbers for pairs are defined in section 3.2. Finally, in section 4, we show our main results (Theorems 4.2 and 4.9).

1.1. Notation and terminology.

1. Let R be a ring, let M be an R -module, and let Γ be a subset of M . We denote by $\langle \Gamma \rangle_R$ the R -submodule of M spanned by Γ .

2. Let X be a projective variety over a field k of characteristic zero. The field of rational functions on X is denoted by $\text{Rat}(X)$. Let \mathbb{K} be either a blank, \mathbb{Q} or \mathbb{R} . The \mathbb{K} -module of all the \mathbb{K} -Cartier divisors (respectively, \mathbb{K} -Weil divisors) on X is denoted by $\text{Div}_{\mathbb{K}}(X)$ (respectively, $\text{WDiv}_{\mathbb{K}}(X)$).

Let D be an \mathbb{R} -Cartier divisor on X , which can be written as

$$D = \sum_{i=1}^l a_i D_i$$

with $D_i \in \text{Div}(X)$ and $a_i \in \mathbb{R}$. A *local equation defining D around a point $x \in X$* is

$$(1.1) \quad f_x := f_1^{\otimes a_1} \otimes \cdots \otimes f_l^{\otimes a_l} \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R},$$

where f_i is a local equation defining D_i around x .

The (*Cartier*) *support* of $D \in \text{Div}_{\mathbb{R}}(X)$ is defined as

$$(1.2) \quad \text{Supp}_{\text{C}}(D) := \{x \in X : f_x \notin \mathcal{O}_{X,x}^* \otimes_{\mathbb{Z}} \mathbb{R}\}.$$

It is known that $\text{Supp}_{\text{C}}(D)$ is a proper Zariski closed subset of X (see [25, Proposition 1.1.1]).

Let $\mu : Y \rightarrow X$ be a morphism of k -varieties. If $\mu(Y)$ is not contained in $\text{Supp}_{\text{C}}(D)$, then one can define the pull-back $\mu^*D \in \text{Div}_{\mathbb{R}}(Y)$.

The (*Weil*) *support* of $D \in \text{WDiv}_{\mathbb{R}}(X)$ is defined as

$$(1.3) \quad \text{Supp}_{\text{W}}(D) := \bigcup_{\substack{Z: \text{ prime Weil divisor,} \\ \text{ord}_Z(D) \neq 0}} Z.$$

It is known that, if X is regular, then

$$\text{Supp}_{\text{C}}(D) = \text{Supp}_{\text{W}}(D)$$

(see for example [25, Proposition 1.1.3]). Actually, this equation is valid as soon as X is normal (see Lemma 2.3(2)), so we can write

$$(1.4) \quad \text{Supp}(D) := \text{Supp}_{\text{C}}(D) = \text{Supp}_{\text{W}}(D).$$

Suppose that X is normal. Let $D \in \text{WDiv}_{\mathbb{R}}(X)$, and let \mathbb{K} be either a blank, \mathbb{Q} , or \mathbb{R} . We set

$$(1.5) \quad H_{\mathbb{K}}^0(D) := \{\phi \in \text{Rat}(X) \otimes_{\mathbb{Z}} \mathbb{K} : D + (\phi) \geq 0\} \cup \{0\}.$$

3. Let $\mu : X' \rightarrow X$ be a morphism of projective varieties over a field. The *exceptional locus* of μ is defined as the minimal Zariski closed subset of X' such that the restriction

$$\mu : X' \setminus \text{Ex}(\mu) \rightarrow X$$

is an immersion (see [19, (3.6)]). If X is normal, then, by Zariski's main theorem [14, Proposition (4.4.1)], one has

$$(1.6) \quad \text{Ex}(\mu) = \{x' \in X' : \dim_{x'}(\mu^{-1}(\mu(x'))) \geq 1\} = \bigcup_{\substack{Z \subset X', \\ \dim \mu(Z) < \dim Z}} Z.$$

4. Let K be a number field, and let O_K be the ring of integers of K . The set of all the finite places of K is denoted by M_K . For each $v \in M_K$, let K_v be the v -adic completion of K , let O_{K_v} be the ring of integers of K_v , and let \tilde{K}_v be the residue field of K_v .

Let X be a projective K -variety. For each $v \in M_K$, we denote by $(X_v^{\text{an}}, \pi_v : X_v^{\text{an}} \rightarrow X_{K_v})$ the Berkovich analytic space associated to $X_{K_v} := X \times_{\text{Spec}(K)} \text{Spec}(K_v)$ and, for $v = \infty$, denote by $(X_\infty^{\text{an}}, \pi_\infty : X_\infty^{\text{an}} \rightarrow X_\mathbb{C})$ the complex analytic space associated to $X_\mathbb{C} := X \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{C})$.

Let D be an \mathbb{R} -Cartier divisor on X , and let $v \in M_K \cup \{\infty\}$. A D -Green function (of continuous type) on X_v^{an} is a continuous map

$$g_v^{\overline{D}} : (X \setminus \text{Supp}_C(D))_v^{\text{an}} \rightarrow \mathbb{R}$$

such that, for each $x \in X_v^{\text{an}}$, $g_v^{\overline{D}} + \log |f_x|_v^2$ extends to a continuous function around x , where f_x is a local equation defining D around $\pi_v(x)$ (see [25]).

Let U be a non-empty open subscheme of $\text{Spec}(O_K)$. A U -model of (X, D) is a couple $(\mathcal{X}_U, \mathcal{D}_U)$ such that \mathcal{X}_U is a reduced, irreducible, projective, and flat U -scheme with a fixed K -isomorphism from X onto the generic fiber $\mathcal{X}_U \times_U \text{Spec}(K)$, and such that \mathcal{D}_U is an \mathbb{R} -Cartier divisor on \mathcal{X}_U satisfying $\mathcal{D}_U|_X = D$.

Let $(\mathcal{X}_U, \mathcal{D}_U)$ be a U -model of (X, D) and let $v \in M_K \cap U$. We define the D -Green function $g_v^{(\mathcal{X}_U, \mathcal{D}_U)}$ associated to $(\mathcal{X}_U, \mathcal{D}_U)$ as

$$(1.7) \quad g_v^{(\mathcal{X}_U, \mathcal{D}_U)}(x) := -\log |f'_x|^2,$$

where f'_x is a local equation defining \mathcal{D}_U around $r_{\mathcal{X}_U}(x)$ and $r_{\mathcal{X}_U} : X_v^{\text{an}} \rightarrow \mathcal{X}_U \times_U \text{Spec}(\tilde{K}_v)$ denotes the reduction map.

Let \mathbb{K} be either a blank, \mathbb{Q} , or \mathbb{R} . An *adelic \mathbb{K} -Cartier divisor* on X is a couple

$$\overline{D} = \left(D, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}}[v] \right)$$

such that,

- D is a \mathbb{K} -Cartier divisor on X ,
- for each $v \in M_K$, $g_v^{\overline{D}}$ is a D -Green function on X_v^{an} ,
- for $v = \infty$, $g_\infty^{\overline{D}}$ is a D -Green function on X_∞^{an} that is invariant under the complex conjugation, and
- there exists a non-empty open subset U of $\text{Spec}(O_K)$ and a U -model $(\mathcal{X}_U, \mathcal{D}_U)$ of (X, D) such that $g_v^{\overline{D}} = g_v^{(\mathcal{X}_U, \mathcal{D}_U)}$ for every $v \in M_K \cap U$.

We call the U -model $(\mathcal{X}_U, \mathcal{D}_U)$ appearing in the above definition a U -model of definition for \overline{D} . We denote by $\widehat{\text{Div}}_{\mathbb{K}}(X)$ the \mathbb{K} -module of all the adelic \mathbb{K} -Cartier divisors on X .

For each $v \in M_K$, $C_v^0(X)$ denotes the \mathbb{R} -vector space of all the \mathbb{R} -valued continuous functions on X_v^{an} , and, for $v = \infty$, $C_\infty^0(X)$ denotes the \mathbb{R} -vector space of all the \mathbb{R} -valued continuous functions on X_∞^{an} that are invariant under the complex conjugation.

(\mathbb{R} -linear equivalence): For $\overline{D}_1, \overline{D}_2 \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, we write $\overline{D}_1 \sim_{\mathbb{R}} \overline{D}_2$ if there exists a $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\overline{D}_1 - \overline{D}_2 = \widehat{(\phi)}$.

(effective): We say that \overline{D} is *effective* if $D \geq 0$ and $g_v^{\overline{D}} \geq 0$ for every $v \in M_K \cup \{\infty\}$.

(strictly effective): Note that, if $D \geq 0$, then

$$\operatorname{ess.\,inf}_{x \in X_\infty^{\text{an}}} g_v^{\overline{D}}(x) > -\infty$$

for every $v \in M_K \cup \{\infty\}$, and

$$\operatorname{ess.\,inf}_{x \in X_\infty^{\text{an}}} g_v^{\overline{D}}(x) \geq 0$$

for all but finitely many $v \in M_K$.

We say that \overline{D} is *strictly effective* if \overline{D} is effective and

$$\operatorname{ess.\,inf}_{x \in X_\infty^{\text{an}}} g_\infty^{\overline{D}}(x) > 0.$$

We write

$$(1.8) \quad \overline{D}_1 \leq \overline{D}_2 \quad (\text{respectively, } \overline{D}_1 < \overline{D}_2)$$

if $\overline{D}_2 - \overline{D}_1$ is effective (respectively, strictly effective).

Let \mathbb{K} be either a blank, \mathbb{Q} , or \mathbb{R} . We set

$$(1.9) \quad \widehat{\Gamma}_{\mathbb{K}}^f(\overline{D}) := \left\{ \phi \in \operatorname{Rat}(X) \otimes_{\mathbb{Z}} \mathbb{K} : D + (\phi) \geq 0 \text{ and } g_v^{\overline{D} + (\widehat{\phi})} \geq 0, \forall v \in M_K \right\} \cup \{0\},$$

$$(1.10) \quad \widehat{\Gamma}_{\mathbb{K}}^{\text{ss}}(\overline{D}) := \left\{ \phi \in \operatorname{Rat}(X) \otimes_{\mathbb{Z}} \mathbb{K} : \overline{D} + (\widehat{\phi}) > 0 \right\} \cup \{0\},$$

and

$$(1.11) \quad \widehat{\Gamma}_{\mathbb{K}}^s(\overline{D}) := \left\{ \phi \in \operatorname{Rat}(X) \otimes_{\mathbb{Z}} \mathbb{K} : \overline{D} + (\widehat{\phi}) \geq 0 \right\} \cup \{0\}.$$

The *arithmetic volume* of \overline{D} is then defined as

$$(1.12) \quad \widehat{\operatorname{vol}}(\overline{D}) := \limsup_{\substack{m \in \mathbb{N}, \\ m \rightarrow +\infty}} \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

5. We use the following positivity notions of adelic \mathbb{R} -Cartier divisors. Let $\overline{A} = \left(A, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{A}}[v] \right)$ be an adelic \mathbb{R} -Cartier divisor on X .

(big): We say that \overline{A} is *big* if $\widehat{\operatorname{vol}}(\overline{A}) > 0$. The cone of all the big adelic \mathbb{R} -Cartier divisors on X is denoted by $\widehat{\operatorname{Big}}_{\mathbb{R}}(X)$.

(pseudo-effective): We say that \overline{A} is *pseudo-effective* if $\widehat{\operatorname{vol}}(\overline{A} + \overline{B}) > 0$ for every $\overline{B} \in \widehat{\operatorname{Big}}_{\mathbb{R}}(X)$. We write $\overline{D}_1 \preceq \overline{D}_2$ if $\overline{D}_2 - \overline{D}_1$ is pseudo-effective.

(nef): \overline{A} is said to be *relatively nef* if A is nef and $g_v^{\overline{A}}$ is semipositive for every $v \in M_K \cup \{\infty\}$ (see [25, section 4.4] for the notion of semipositivity). \overline{A} is said to be *nef* if \overline{A} is relatively nef and

$$\inf_{x \in X(\overline{K})} h_{\overline{A}}(x) \geq 0,$$

where

$$h_{\overline{A}}(x) := \frac{1}{[k(x) : K]} \widehat{\deg}(\overline{A}|_x)$$

is the *height* of $x \in X(\overline{K})$ with respect to \overline{A} , $k(x)$ is the residue field of the image of x , and

$$\widehat{\deg}(\overline{A}|_x) := \frac{1}{2} \sum_{v \in M_K} \sum_{\substack{w \in M_{k(x)}, \\ w|v}} [k(x)_w : K_v] g_v^{\overline{A}}(x^w) + \frac{1}{2} \sum_{\sigma: k(x) \rightarrow \mathbb{C}} g_{\infty}^{\overline{A}}(x^{\sigma})$$

(see [25, sections 2.4 and 4.2] for detail). We denote by $\widehat{\text{Nef}}_{\mathbb{R}}(X)$ the cone of all the nef adelic \mathbb{R} -Cartier divisors on X . Note that a relatively nef adelic \mathbb{R} -Cartier divisor \overline{A} is nef if and only if

$$\widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y+1)}) \geq 0$$

for every closed subvariety Y of X .

(integrable): \overline{A} is said to be *integrable* if \overline{A} is written as a difference of two nef adelic \mathbb{R} -Cartier divisors. We denote by $\widehat{\text{Int}}_{\mathbb{R}}(X)$ the \mathbb{R} -vector space of all the integrable adelic \mathbb{R} -Cartier divisors on X .

(w-ample): \overline{A} is said to be *weakly ample* or *w-ample* for short if \overline{A} is a positive \mathbb{R} -linear combination $\sum_{i=1}^l a_i \overline{A}_i$ of adelic Cartier divisors \overline{A}_i such that each A_i is ample and such that $H^0(mA_i) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}_i) \rangle_K$ for every $m \gg 1$. Note that this definition does not depend on the choice of K (see [19, Theorem 4.3]).

(ample): \overline{A} is said to be *ample* (in the sense of Zhang) if A is relatively nef and

$$\widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y+1)}) > 0$$

for every closed subvariety Y of X .

Let \mathcal{X} be a normal projective arithmetic variety over $\text{Spec}(O_K)$ such that \mathcal{X}_K is K -isomorphic to X . To an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}} = (\mathcal{D}, g^{\overline{\mathcal{D}}})$ on \mathcal{X} as in [24, section 5], one can associate an adelic \mathbb{R} -Cartier divisor

$$(1.13) \quad \overline{\mathcal{D}}^{\text{ad}} := \left(\mathcal{D}_K, \sum_{v \in M_K} g_v^{(\mathcal{X}, \mathcal{D})}[v] + g^{\overline{\mathcal{D}}}[\infty] \right)$$

on X . We say that $\overline{\mathcal{D}}$ is *w-ample* (respectively, *ample*, etc.) if so is $\overline{\mathcal{D}}^{\text{ad}}$.

6. By [25, section 4.5] and the same arguments as in [18, Lemma 2.5], we can uniquely extend the arithmetic intersection numbers of C^∞ -Hermitian line bundles to a multilinear map

$$(1.14) \quad \widehat{\deg} : \widehat{\text{Int}}_{\mathbb{R}}(X)^{\times \dim X} \times \widehat{\text{Div}}_{\mathbb{R}}(X) \rightarrow \mathbb{R},$$

$$(\overline{D}_1, \dots, \overline{D}_{\dim X+1}) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{\dim X+1}),$$

in such a way that

- (i) the restriction $\widehat{\deg} : \widehat{\text{Int}}_{\mathbb{R}}(X)^{\times(\dim X+1)} \rightarrow \mathbb{R}$ is symmetric,
- (ii) $\widehat{\deg}(\overline{N}^{(\dim X+1)}) = \widehat{\text{vol}}(\overline{N})$ for every $\overline{N} \in \widehat{\text{Nef}}_{\mathbb{R}}(X)$, and
- (iii) $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{\dim X+1}) \geq 0$ for every $\overline{D}_1, \dots, \overline{D}_{\dim X} \in \widehat{\text{Nef}}_{\mathbb{R}}(X)$ and $\overline{D}_{\dim X+1} \succeq 0$.

2. ADELIC CARTIER DIVISORS WITH BASE CONDITIONS

2.1. Preliminaries on the valuations. In this subsection, we recall some basic facts on the valuations.

Definition 2.1. Let (Λ, \leq) be a totally ordered \mathbb{Z} -module of rank r . By [6, Chap. VI, §10, no. 2, Proposition 4], (Λ, \leq) is isomorphic to $(\mathbb{Z}^r, \leq_{\text{lex}})$ if and only if the height of Λ equals r , where \leq_{lex} denotes the lexicographic order.

Let $F \supset k$ be a field extension and let

$$(2.1) \quad \nu : F^\times \rightarrow \Lambda$$

be a *valuation of F/k with values in Λ* , namely, ν satisfies

- (i) $\nu(a) = 0$ for $a \in k^\times$,
- (ii) $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ for $\phi, \psi \in F^\times$, and
- (iii) $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for $\phi, \psi \in F^\times$ with $\phi + \psi \neq 0$.

The *valuation ring* of ν is

$$(2.2) \quad O_\nu := \{\phi \in F^\times : \nu(\phi) \geq 0\} \cup \{0\}$$

and the *maximal ideal* of ν is

$$(2.3) \quad \mathfrak{m}_\nu := \{\phi \in F^\times : \nu(\phi) > 0\} \cup \{0\}.$$

We put $O_\nu^* := O_\nu \setminus \mathfrak{m}_\nu = \{\phi \in F^\times : \nu(\phi) = 0\}$, and put $k_\nu := O_\nu / \mathfrak{m}_\nu$. The *value group* of ν is defined as

$$(2.4) \quad \Lambda_\nu := \nu(F^\times) = F^\times / O_\nu^*$$

endowed with the order \leq , and the *rational rank* of ν is defined as $\text{rat.rk}(\nu) := \text{rk}_{\mathbb{Q}} \Lambda_\nu \otimes_{\mathbb{Z}} \mathbb{Q}$.

Two valuations $\nu_1 : F^\times \rightarrow \Lambda_{\nu_1}$ and $\nu_2 : F^\times \rightarrow \Lambda_{\nu_2}$ are said to be *equivalent* if the following equivalent conditions are satisfied.

- There exists an order-preserving isomorphism $\iota : \Lambda_{\nu_1} \rightarrow \Lambda_{\nu_2}$ such that $\nu_2 = \iota \circ \nu_1$.
- $O_{\nu_1} = O_{\nu_2}$ in F .

A valuation $\nu : F^\times \rightarrow \Lambda_\nu$ is said to be *(non-trivial) discrete* if the value group (Λ_ν, \leq) is isomorphic to (\mathbb{Z}, \leq) . We denote by $\mathfrak{V}(F) = \mathfrak{V}(F/k)$ the set of all the equivalence classes of the discrete valuations of F/k . Given any $\nu \in \mathfrak{V}(F)$, there exists a unique valuation ν' of F such that ν' is equivalent to ν and ν' has value group (\mathbb{Z}, \leq) . So, in the following, we always assume that the value group of a $\nu \in \mathfrak{V}(F)$ is *normalized* to (\mathbb{Z}, \leq) .

Definition 2.2. Let X be a projective variety over a field k , let $\text{Rat}(X)$ be the field of rational functions on X , and let $\nu : \text{Rat}(X)^\times \rightarrow \Lambda_\nu$ be a valuation of $\text{Rat}(X)/k$. The *center of ν on X* is a point $c_X(\nu) \in X$ such that $\mathcal{O}_{X, c_X(\nu)} \subset O_\nu$ and $\mathfrak{m}_{c_X(\nu)} = \mathfrak{m}_\nu \cap \mathcal{O}_{X, c_X(\nu)}$. By the valuative criterion of properness, there exists a unique center on X for any valuation ν .

Let D be an effective Cartier divisor on X . If f, g are two local equations defining D around $c_X(\nu)$, then $f/g \in \mathcal{O}_{X, c_X(\nu)}^*$. So $\nu(f) = \nu(g)$, and we can set

$$(2.5) \quad \nu_X(D) := \nu(f) \in \Lambda_\nu.$$

Note that $\nu_X(0)$ is defined as $\nu(1) = 0$. Since

$$\nu_X(D + D') = \nu_X(D) + \nu_X(D')$$

for two effective Cartier divisors D, D' on X , we can uniquely extend, by linearity, the map $\nu_X : D \mapsto \nu_X(D)$ to an \mathbb{R} -linear map

$$(2.6) \quad \nu_X : \text{Div}_{\mathbb{R}}(X) \rightarrow \Lambda_{\nu} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 2.1. Let $\pi : X' \rightarrow X$ be a birational projective morphism. Then

$$\nu_{X'}(\pi^*D) = \nu_X(D).$$

In fact, we have $\pi(c_{X'}(\nu)) = c_X(\nu)$. So, if f is a local equation defining D around $c_X(\nu)$, then π^*f is a local equation defining π^*D around $c_{X'}(\nu)$. Hence $\nu_{X'}(\pi^*D) = \nu(\pi^*f) = \nu(f) = \nu_X(D)$.

Lemma 2.2. *Let X be a projective variety over a field and let D be an \mathbb{R} -Cartier divisor on X .*

- (1) *There exist Cartier divisors D_1, \dots, D_l and $a_1, \dots, a_l \in \mathbb{R}$ such that a_1, \dots, a_l are \mathbb{Q} -linearly independent and $D = \sum_{i=1}^l a_i D_i$. Moreover, in this case, one has*

$$\text{Supp}_{\mathbb{C}}(D) = \bigcup_{i=1}^l \text{Supp}_{\mathbb{C}}(D_i) \quad \text{and} \quad \text{Supp}_{\mathbb{W}}(D) = \bigcup_{i=1}^l \text{Supp}_{\mathbb{W}}(D_i).$$

- (2) *Let $\phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$. There exist $\phi_1, \dots, \phi_l \in \text{Rat}(X)^{\times}$ and $a_1, \dots, a_l \in \mathbb{R}$ such that a_1, \dots, a_l are \mathbb{Q} -linearly independent and $(\phi) = \sum_{i=1}^l a_i (\phi_i)$. Moreover, in this case, one has*

$$\text{Supp}_{\mathbb{C}}((\phi)) = \bigcup_{i=1}^l \text{Supp}_{\mathbb{C}}((\phi_i)) \quad \text{and} \quad \text{Supp}_{\mathbb{W}}((\phi)) = \bigcup_{i=1}^l \text{Supp}_{\mathbb{W}}((\phi_i)).$$

Proof. (1): Choose an expression $D = \sum_{i=1}^l a_i D_i$ such that $a_i \in \mathbb{R}$, $D_i \in \text{Div}_{\mathbb{Q}}(X)$, and l is minimal among such expressions. If a_i are \mathbb{Q} -linearly dependent, one can find, after renumbering a_1, \dots, a_l , an expression

$$\sum_{i=1}^l r_i a_i = 0$$

such that $r_1, \dots, r_l \in \mathbb{Z}$ and $r_l \neq 0$. Then

$$D = \sum_{i=1}^{l-1} a_i D_i - \frac{1}{r_l} \left(\sum_{i=1}^{l-1} r_i a_i \right) D_l = \sum_{i=1}^{l-1} \frac{a_i}{r_l} (r_l D_i - r_i D_l),$$

which contradicts the minimality of l .

For the first equality, the inclusion \subset is clear. Suppose that $x \notin \text{Supp}_{\mathbb{C}}(D)$, so that

$$f_1^{\otimes a_1} \otimes \dots \otimes f_l^{\otimes a_l} \in \mathcal{O}_{X,x}^* \otimes_{\mathbb{Z}} \mathbb{R},$$

where f_i is a local equation defining D_i around x . By applying [25, Lemma 1.3.1], we have $f_i \in \mathcal{O}_{X,x}^* \otimes_{\mathbb{Z}} \mathbb{Q}$ for every i . Thus $x \notin \text{Supp}_{\mathbb{C}}(D_i)$ for every i .

For the second, the inclusion \subset is clear. Let V be the \mathbb{Q} -subspace of $\text{WDiv}_{\mathbb{Q}}(X)$ generated by the irreducible components of $\text{Supp}_{\mathbb{W}}(D)$. Since $a_1 D_1 + \dots + a_l D_l \in V \otimes_{\mathbb{Q}} \mathbb{R}$, we have, by [25, Lemma 1.3.1], $D_i \in V$ for every i . So $\text{Supp}_{\mathbb{W}}(D_i) \subset \text{Supp}_{\mathbb{W}}(D)$ for every i .

By the same arguments, we can also show the assertion (2). \square

Lemma 2.3. *Let $D \in \text{Div}_{\mathbb{R}}(X)$, and suppose that X is normal.*

- (1) If $D \geq 0$, then $\nu_X(D) \geq 0$.
 (2) One has

$$\text{Supp}_C(D) = \text{Supp}_W(D) = \bigcup_{\substack{\nu \in \mathfrak{V}(\text{Rat}(X)), \\ \nu_X(D) \neq 0}} \overline{\{c_X(\nu)\}}.$$

Proof. (1): Let $\pi : X' \rightarrow X$ be a resolution of singularities, and write

$$\pi^* D = a_1 D'_1 + \cdots + a_l D'_l$$

with $a_i \geq 0$ and prime divisors D'_i on X' . Let f_i be a local equation defining D'_i around $c_{X'}(\nu)$. Since $f_i \in \mathcal{O}_{X', c_{X'}(\nu)} \setminus \{0\}$, one has $\nu(f_i) \geq 0$ and

$$\nu_X(D) = \nu_{X'}(\pi^* D) = \sum_{i=1}^l a_i \nu(f_i) \geq 0.$$

(2): Let $\nu \in \mathfrak{V}(\text{Rat}(X))$ such that $\nu_X(D) \neq 0$. If $c_X(\nu) \notin \text{Supp}_C(D)$, there exists a local equation $f \in \mathcal{O}_{X, c_X(\nu)}^* \otimes_{\mathbb{Z}} \mathbb{R}$ that defines D around $c_X(\nu)$. So $\nu_X(D) = \nu(f) = 0$, and it is a contradiction. As a result, we have inclusions

$$\text{Supp}_C(D) \supset \bigcup_{\substack{\nu \in \mathfrak{V}(\text{Rat}(X)), \\ \nu_X(D) \neq 0}} \overline{\{c_X(\nu)\}} \supset \text{Supp}_W(D).$$

We are going to show $\text{Supp}_C(D) = \text{Supp}_W(D)$. Let $D = \sum_{i=1}^l a_i D_i$ be an expression as in Lemma 2.2(1). Since

$$\text{Supp}_C(D) = \bigcup_{i=1}^l \text{Supp}_C(D_i) \quad \text{and} \quad \text{Supp}_W(D) = \bigcup_{i=1}^l \text{Supp}_W(D_i),$$

we can assume $D \in \text{Div}(X)$. In this case, $\text{Supp}_C(D)$ is nothing but the usual (Cartier) support $\{x \in X : f \notin \mathcal{O}_{X, c_X(\nu)}^*\}$ by [25, Proposition 1.1.1].

We endow $\text{Supp}_C(D)$ with the reduced induced scheme structure, and let x be a maximal point of $\text{Supp}_C(D)$. By [15, Corollaire (21.1.9)], one has $\text{depth}(\mathcal{O}_{X, x}) = 1$, so $\dim(\mathcal{O}_{X, x}) = 1$ since X is normal. Therefore, $\text{Supp}_C(D)$ is a Zariski closed subset of pure codimension one in X .

Let $\pi : X' \rightarrow X$ be a resolution of singularities of X . There exists an open subset U of X such that $\text{codim}(X \setminus U, X) \geq 2$ and $\pi : \pi^{-1}(U) \rightarrow U$ is an isomorphism. If $x \in U$ is a maximal point of $\text{Supp}_C(D)$, then $\mathcal{O}_{X', \pi^{-1}(x)} = \mathcal{O}_{X, x}$, and $\pi^{-1}(x)$ belongs to $\text{Supp}_C(\pi^* D) = \text{Supp}_W(\pi^* D)$. Clearly, $\pi(\text{Supp}_W(\pi^* D)) = \text{Supp}_W(D)$, so $x \in \text{Supp}_W(D)$. \square

Lemma 2.4. *Let $F \supset k$ be a field extension.*

- (1) *Let Λ, Λ' be totally ordered \mathbb{Z} -modules. Let $\ell : \Lambda \rightarrow \Lambda'$ be an order-preserving homomorphism, namely, ℓ is a homomorphism of \mathbb{Z} -modules such that $\lambda \geq 0$ implies $\ell(\lambda) \geq 0$ for every $\lambda \in \Lambda$. If $\nu : F^\times \rightarrow \Lambda$ is a valuation of F , then so is $\ell \circ \nu : F^\times \rightarrow \Lambda'$.*
 (2) *Suppose that F is finitely generated over k . Let X be a projective k -variety with $\text{Rat}(X) = F$ and let $r := \text{tr.deg}_k F$. For a $\nu \in \mathfrak{V}(F)$, the following are equivalent.*
 (a) $\text{tr.deg}_k k_\nu = r - 1$.

- (b) There exists a valuation $\nu : F^\times \rightarrow \mathbb{Z}^r$ of F such that ν has value group $(\mathbb{Z}^r, \leq_{\text{lex}})$ and $\nu = \text{pr}_1 \circ \nu$.
- (c) There exist a valuation $\nu : F^\times \rightarrow \Lambda_\nu$ of F and an order-preserving surjective linear form $\ell : \Lambda_\nu \rightarrow \mathbb{Z}$ such that $\text{rk}_{\mathbb{Q}} \Lambda_\nu \otimes_{\mathbb{Z}} \mathbb{Q} = r$ and $\nu = \ell \circ \nu$.
- (d) There exist a birational projective morphism $X' \rightarrow X$ and a prime Weil divisor Y' of X' such that X' is normal and ν is equivalent to $\text{ord}_{Y'}$.

Definition 2.3. A valuation $\nu : \text{Rat}(X)^\times \rightarrow \Lambda_\nu$ is called *divisorial* if ν is discrete and the equivalent conditions in Lemma 2.4(2) are satisfied.

Proof of Lemma 2.4. The assertion (1) is obvious.

(2): Since ν is a discrete valuation, \mathfrak{m}_ν is a principal ideal with generator ϖ . The implication (b) \Rightarrow (c) is clear.

(a) \Rightarrow (b): The valuation ν satisfies $\text{rat.rk}(\nu) = 1$ and

$$\text{rat.rk}(\nu) + \text{tr.deg}_k k_\nu = \text{tr.deg}_k F.$$

So, by [6, Chap. VI, §10, no. 3, Corollaire 1], k_ν is a finitely generated field extension of k with $\text{tr.deg}_k k_\nu = r - 1$. There exists a valuation $\bar{\nu} : k_\nu^\times \rightarrow \mathbb{Z}^{r-1}$ having value group $(\mathbb{Z}^{r-1}, \leq_{\text{lex}})$. We define $\nu : F^\times \rightarrow \mathbb{Z}^r$ by

$$\phi \mapsto (\nu(\phi), \bar{\nu}(\phi \cdot \varpi^{-\nu(\phi)} \bmod \mathfrak{m}_\nu)).$$

One can verify that ν is a valuation of F . Given any $(n_1, \dots, n_r) \in \mathbb{Z}^r$, there exists a $\psi \in O_\nu^*$ such that $\bar{\nu}(\psi \bmod \mathfrak{m}_\nu) = (n_2, \dots, n_r)$. So $\nu(\psi \cdot \varpi^{n_1}) = (n_1, \dots, n_r)$ and the value group of ν is $(\mathbb{Z}^r, \leq_{\text{lex}})$.

(c) \Rightarrow (a): We know that \mathfrak{m}_ν is a prime ideal of O_ν and $\bar{V} := O_\nu / \mathfrak{m}_\nu$ is a valuation ring for k_ν (see [6, Chap. VI, §4, no. 1, Propositions 1 et 2]). So we have a homomorphism of semigroups

$$\bar{V} \setminus \{0\} = O_\nu \setminus \mathfrak{m}_\nu \xrightarrow{\nu} \text{Ker}(\ell).$$

We can uniquely extend this to a homomorphism $\bar{\nu} : k_\nu^\times \rightarrow \text{Ker}(\ell)$ of abelian groups. We are going to show that $\bar{\nu}$ is a valuation of k_ν with $\text{rat.rk}(\bar{\nu}) = r - 1$. The conditions (i) and (ii) are obvious. For $\phi, \psi \in O_\nu \setminus \mathfrak{m}_\nu$ with $\phi + \psi \notin \mathfrak{m}_\nu$, we have

$$\begin{aligned} \bar{\nu}(\phi + \psi \bmod \mathfrak{m}_\nu) &= \nu(\phi + \psi) \\ &\geq \min\{\nu(\phi), \nu(\psi)\} = \min\{\bar{\nu}(\phi \bmod \mathfrak{m}_\nu), \bar{\nu}(\psi \bmod \mathfrak{m}_\nu)\}. \end{aligned}$$

So the condition (iii) holds in general. Take an $e_1 \in \Lambda_\nu$ with $\ell(e_1) = 1$. Then $\ell : \Lambda_\nu \rightarrow \mathbb{Z}$ splits and $\Lambda_\nu = \mathbb{Z}e_1 \oplus \text{Ker}(\ell)$ as ordered \mathbb{Z} -modules, where the right-hand side is endowed with the lexicographic order. Let $\lambda \in \text{Ker}(\ell)$. Either λ or $-\lambda$ is non-negative, so we may assume $\lambda \geq 0$. There exists a $\phi \in O_\nu$ such that $\nu(\phi) = \lambda \geq 0$, so $\bar{\nu}$ is surjective. Since $\text{rat.rk}(\bar{\nu}) \leq \text{tr.deg}_k k_\nu \leq r - 1$, we have $\text{tr.deg}_k k_\nu = r - 1$.

(a) \Rightarrow (d): By [27, Proposition 2.3], there exist a birational projective morphism $X' \rightarrow X$ and a point $\xi' \in X'$ of codimension one such that X' is normal and ν has center ξ' on X' . Since $\mathcal{O}_{X', \xi'}$ is a discrete valuation ring dominated by O_ν , we have $\mathcal{O}_{X', \xi'} = O_\nu$ and ν is equivalent to $\text{ord}_{\xi'}$.

(d) \Rightarrow (a): If ν is equivalent to $\text{ord}_{Y'}$, then $k_\nu = \text{Rat}(Y')$ has transcendence degree $r - 1$ over k . \square

2.2. Base conditions. A purpose of this subsection is to introduce the notion of pairs of adelic \mathbb{R} -Cartier divisors and base conditions (see Definition 2.5).

Definition 2.4. Let K be a number field, let X be a projective K -variety, and let $\mathfrak{V}(\text{Rat}(X))$ be the set of all the normalized (non-trivial) discrete valuations of $\text{Rat}(X)/K$. Let \mathbb{K} be either \mathbb{R} , \mathbb{Q} , or \mathbb{Z} . A \mathbb{K} -base condition is defined as a finite formal sum

$$(2.7) \quad \mathcal{V} := \sum_{\nu \in \mathfrak{V}(\text{Rat}(X))} a_\nu [\nu],$$

where $a_\nu \in \mathbb{K}$ and $\nu \in \mathfrak{V}(\text{Rat}(X))$. We denote by $\text{BC}_{\mathbb{K}}(X)$ the \mathbb{K} -module of all the \mathbb{K} -base conditions on X . The *order* of an \mathbb{R} -base condition \mathcal{V} along ν is defined as

$$(2.8) \quad \nu(\mathcal{V}) := a_\nu.$$

We say that \mathcal{V} is *effective* if $\nu(\mathcal{V}) \geq 0$ for every $\nu \in \mathfrak{V}(\text{Rat}(X))$ and denote it by $\mathcal{V} \geq 0$. Put

$$(2.9) \quad \mathcal{V}_+ := \sum_{\nu(\mathcal{V}) \geq 0} \nu(\mathcal{V}) [\nu] \quad \text{and} \quad \mathcal{V}_- := \mathcal{V}_+ - \mathcal{V}.$$

We say that \mathcal{V} is *divisorial* if $\nu(\mathcal{V}) \neq 0$ implies that ν is divisorial (see Definition 2.3).

We define the *support* on X of an \mathbb{R} -base condition \mathcal{V} as

$$(2.10) \quad \text{Supp}_X(\mathcal{V}) := \bigcup_{\nu(\mathcal{V}) \neq 0} \overline{\{c_X(\nu)\}},$$

which is a Zariski closed subset of X .

We can naturally regard an \mathbb{R} -Cartier divisor as an \mathbb{R} -Weil divisor. To an \mathbb{R} -Weil divisor

$$\Xi = \sum_{Z: \text{prime Weil divisor}} a_Z Z,$$

we can naturally associate an \mathbb{R} -base condition

$$(2.11) \quad [\Xi] := \sum_{Z: \text{prime Weil divisor}} a_Z [\text{ord}_Z].$$

Remark 2.5. Let Z be a prime Weil divisor on X , and let $D \in \text{Div}_{\mathbb{R}}(X)$. One then has

$$\text{ord}_Z([D]) = \text{ord}_{Z,X}(D)$$

(see (2.6), (2.8), and (2.11)). In fact, it suffices to show the equality for $D \in \text{Div}(X)$. Let f be a local equation defining D around $c_X(\nu)$. Then $\text{ord}_{Z,X}(D) = \text{ord}_Z(f) = \text{ord}_Z([D])$.

Definition 2.5. Let \mathbb{K} and \mathbb{K}' be either \mathbb{R} , \mathbb{Q} , or \mathbb{Z} . We define

$$(2.12) \quad \widehat{\text{BDiv}}_{\mathbb{K},\mathbb{K}'}(X) := \widehat{\text{Div}}_{\mathbb{K}}(X) \times \text{BC}_{\mathbb{K}'}(X),$$

$$(2.13) \quad \widehat{\text{WDiv}}_{\mathbb{K},\mathbb{K}'}(X) := \widehat{\text{Div}}_{\mathbb{K}}(X) \times \text{WDiv}_{\mathbb{K}'}(X),$$

$$(2.14) \quad \widehat{\text{Div}}_{\mathbb{K},\mathbb{K}'}(X) := \widehat{\text{Div}}_{\mathbb{K}}(X) \times \text{Div}_{\mathbb{K}'}(X),$$

and

$$(2.15) \quad \widehat{\text{DDiv}}_{\mathbb{K},\mathbb{K}'}(X) := \left\{ (\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{K},\mathbb{K}'}(X) : \mathcal{V} \text{ is divisorial} \right\}$$

(see Definition 2.4 and Notation and terminology 2). If the base condition \mathcal{V} is zero, we always identify a pair $(\overline{D}; 0)$ with the adelic \mathbb{R} -Cartier divisor \overline{D} . So we have natural inclusions of four types of the base conditions;

$$\widehat{\text{Div}}_{\mathbb{K}}(X) \subset \widehat{\text{Div}}_{\mathbb{K}, \mathbb{K}'}(X) \subset \widehat{\text{WDiv}}_{\mathbb{K}, \mathbb{K}'}(X) \subset \widehat{\text{DDiv}}_{\mathbb{K}, \mathbb{K}'}(X) \subset \widehat{\text{BDiv}}_{\mathbb{K}, \mathbb{K}'}(X).$$

Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$.

(effective): We say that $(\overline{D}; \mathcal{V})$ is *effective* (respectively, *strictly effective*) if $\overline{D} \geq 0$ (respectively, $\overline{D} > 0$; see Notation and terminology 4 for definition of inequality signs) and $\nu_X(D) \geq \nu(\mathcal{V})$ for every $\nu \in \mathfrak{V}(\text{Rat}(X))$. For two pairs $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)$ on X , we write

$$(2.16) \quad (\overline{D}_1; \mathcal{V}_1) \leq (\overline{D}_2; \mathcal{V}_2) \quad (\text{respectively, } (\overline{D}_1; \mathcal{V}_1) < (\overline{D}_2; \mathcal{V}_2))$$

if $(\overline{D}_2 - \overline{D}_1; \mathcal{V}_2 - \mathcal{V}_1)$ is effective (respectively, strictly effective).

(ν_0 -effective): Let $\nu_0 \in \mathfrak{V}(\text{Rat}(X))$. We say that $(\overline{D}; \mathcal{V})$ is ν_0 -*effective* (respectively, *strictly ν_0 -effective*) if $(\overline{D}; \mathcal{V}) \geq 0$ (respectively, $(\overline{D}; \mathcal{V}) > 0$) and $\nu_{0,X}(D) = \nu_0(\mathcal{V})$. We write

$$(2.17) \quad (\overline{D}_1; \mathcal{V}_1) \leq_{\nu_0} (\overline{D}_2; \mathcal{V}_2) \quad (\text{respectively, } (\overline{D}_1; \mathcal{V}_1) <_{\nu_0} (\overline{D}_2; \mathcal{V}_2))$$

if $(\overline{D}_2 - \overline{D}_1; \mathcal{V}_2 - \mathcal{V}_1)$ is ν_0 -effective (respectively, strictly ν_0 -effective). Obviously, if $(\overline{D}; \mathcal{V}) \geq_{\nu_0} 0$, then $\nu_0(\mathcal{V}) = \nu_{0,X}(D) \geq 0$.

Let \mathbb{K} be either a blank, \mathbb{Q} , or \mathbb{R} . Given a pair $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, we set

$$(2.18) \quad \widehat{\Gamma}_{\mathbb{K}}^{\text{ss}}(\overline{D}; \mathcal{V}) := \left\{ \phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K} : (\overline{D} + \widehat{\phi}; \mathcal{V}) > 0 \right\} \cup \{0\}$$

and

$$(2.19) \quad \widehat{\Gamma}_{\mathbb{K}}^{\text{s}}(\overline{D}; \mathcal{V}) := \left\{ \phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K} : (\overline{D} + \widehat{\phi}; \mathcal{V}) \geq 0 \right\} \cup \{0\}$$

(see Definition 2.5 for definition of inequality signs). If Ξ is an effective \mathbb{R} -Weil divisor on X , then

$$\widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; [\Xi]) = \widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}) \cap H_{\mathbb{K}}^0(D - \Xi)$$

for $? = \text{s}$ or ss and $\mathbb{K} = \text{a blank, } \mathbb{Q}, \text{ or } \mathbb{R}$.

It follows from definition that

$$(2.20) \quad (\overline{E}; [E]) \geq_{\nu} 0$$

for every effective $\overline{E} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ and $\nu \in \mathfrak{V}(\text{Rat}(X))$. Moreover,

$$(2.21) \quad (0; -\mathcal{V}) \geq 0 \quad (\text{respectively, } (0; -\mathcal{V}) \geq_{\nu} 0)$$

for every $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ with $\mathcal{V} \geq 0$ (respectively, $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ with $\mathcal{V} \geq 0$ and $\nu(\mathcal{V}) = 0$).

Remark 2.6. (1) By Lemma 2.3(1), it follows that $\widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V}) = \widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V}_+)$ for $? = \text{s}$ or ss and $\mathbb{K} = \text{a blank, } \mathbb{Q}, \text{ or } \mathbb{R}$.

(2) Let \mathbb{K} be either a blank, \mathbb{Q} , or \mathbb{R} . Let $(\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}') \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$. If $\phi \in \widehat{\Gamma}_{\mathbb{K}}^{\text{ss}}(\overline{D}; \mathcal{V})$ and $\phi' \in \widehat{\Gamma}_{\mathbb{K}}^{\text{s}}(\overline{D}'; \mathcal{V}')$, then $\phi \cdot \phi' \in \widehat{\Gamma}_{\mathbb{K}}^{\text{ss}}(\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}')$. Here, \mathcal{V} and \mathcal{V}' may not be effective.

Lemma 2.7. *Let S be a projective scheme over a Noetherian ring and let \mathcal{A} be an ample invertible sheaf on S . Let T be a closed subscheme of S and let $x \in S$ be a point not contained in T . There exist an $m \geq 1$ and an $s \in H^0(\mathcal{A}^{\otimes m})$ such that s vanishes along T and $s(x) \neq 0$.*

Proof. Let \mathcal{I} be the ideal sheaf defining T . For an $m \geq 1$, $\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m}$ is generated by its global sections. Since $x \notin T$, $(\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m})_x$ is isomorphic to $\mathcal{O}_{S,x}$ as $\mathcal{O}_{S,x}$ -modules. So there exists an $s' \in H^0(\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m})$ such that $s'(x) \neq 0$. The image of s' via $H^0(\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m}) \rightarrow H^0(\mathcal{A}^{\otimes m})$ has the required properties. \square

The following lemma, which we will use in the proof of Theorem 2.21, gives a sufficient condition to generalize the relation (2.20).

Lemma 2.8. *Let $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ and let $\nu_0 \in \mathfrak{V}(\text{Rat}(X))$.*

- (1) *There exists an adelic Cartier divisor \overline{A} such that $(\overline{A}; \mathcal{V}) > 0$.*
- (2) *For $\nu \in \mathfrak{V}(\text{Rat}(X))$, the following are equivalent.*
 - (a) *There exists an adelic \mathbb{Q} -Cartier divisor \overline{A} such that $(\overline{A}; [\nu]) >_{\nu_0} 0$.*
 - (b) *Either $\nu = \nu_0$ or $c_X(\nu_0) \notin \overline{\{c_X(\nu)\}}$.*
- (3) *If*

$$\nu_0(\mathcal{V}) \geq 0 \quad \text{and} \quad c_X(\nu_0) \notin \text{Supp}_X(\mathcal{V}_+ - \nu_0(\mathcal{V})[\nu_0]),$$

then there exists an adelic \mathbb{R} -Cartier divisor \overline{A} such that $(\overline{A}; \mathcal{V}) >_{\nu_0} 0$.

Proof. We omit the proof of the assertion (1).

(2)(a) \Rightarrow (b): Assume $\nu \neq \nu_0$ in $\mathfrak{V}(\text{Rat}(X))$. Since $(\overline{A}; [\nu]) >_{\nu_0} 0$, $A \geq 0$ and $\nu_{0,X}(A) = 0$, so $c_X(\nu_0) \notin \text{Supp}(A)$. On the other hand, since $\nu_X(A) = 1$, one has $c_X(\nu) \in \text{Supp}(A)$. Hence, $c_X(\nu_0) \notin \overline{\{c_X(\nu)\}}$.

(b) \Rightarrow (a): Choose a strictly effective adelic Cartier divisor \overline{A}'_{ν_0} such that A'_{ν_0} passes through $c_X(\nu_0)$, and set $\overline{A}_{\nu_0} := (1/\nu_{0,X}(A'_{\nu_0}))\overline{A}'_{\nu_0}$. Then

$$(\overline{A}_{\nu_0}; [\nu_0]) >_{\nu_0} 0.$$

Let $\nu \in \mathfrak{V}(\text{Rat}(X))$ such that $c_X(\nu_0) \notin \overline{\{c_X(\nu)\}}$. By Lemma 2.7, there exists an effective Cartier divisor A'_ν on X such that

$$c_X(\nu) \in \text{Supp}(A'_\nu) \quad \text{and} \quad c_X(\nu_0) \notin \text{Supp}(A'_\nu).$$

We endow $A_\nu := (1/\nu_X(A'_\nu))A'_\nu$ with A_ν -Green functions such that $\overline{A}_\nu > 0$. We then have

$$(\overline{A}_\nu; [\nu]) >_{\nu_0} 0.$$

(3): Since $\nu_0(\mathcal{V}_-) = 0$, we have

$$(\overline{\mathcal{D}}; \mathcal{V}) \geq_{\nu_0} (\overline{\mathcal{D}}; \mathcal{V}_+)$$

for every adelic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ (see (2.21)), so that we can assume $\mathcal{V} \geq 0$. We fix \overline{A}_{ν_0} and \overline{A}_ν as above for $\nu \in \mathfrak{V}(\text{Rat}(X))$ with $c_X(\nu_0) \notin \overline{\{c_X(\nu)\}}$, and set

$$\overline{A} := \nu_0(\mathcal{V})\overline{A}_{\nu_0} + \sum_{c_X(\nu_0) \notin \overline{\{c_X(\nu)\}}} \nu(\mathcal{V})\overline{A}_\nu.$$

Then $(\overline{A}; \mathcal{V}) >_{\nu_0} 0$. \square

Definition 2.6. Let X be a projective K -variety and let

$$\overline{D} = \left(D, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}} \right)$$

be an adelic \mathbb{R} -Cartier divisor on X . By Lemma 2.2(1), one can write

$$D = \sum_{i=1}^l a_i D_i$$

with Cartier divisors D_1, \dots, D_l and $a_i \in \mathbb{R}$ such that $\text{Supp}(D) = \bigcup_{i=1}^l \text{Supp}(D_i)$. Let $\iota : Y \rightarrow X$ be a K -morphism of projective K -varieties. If $\iota(Y)$ is not contained in $\text{Supp}(D)$, then $\iota(Y)$ is not contained in $\text{Supp}(D_i)$ for every i . One can define the *pull-back* of \overline{D} via ι by

$$\iota^* \overline{D} := \left(\sum_{i=1}^l a_i \iota^* D_i, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}} \circ \iota_v^{\text{an}}[v] \right),$$

which one can see is actually an adelic \mathbb{R} -Cartier divisor on Y (see [25, Proposition 2.1.4]).

Let \overline{D}' be another adelic \mathbb{R} -Cartier divisor on X such that $\overline{D}' \sim_{\mathbb{R}} \overline{D}$ and $\iota(Y)$ is not contained in $\text{Supp}(D')$. By using Lemma 2.2(1), one can find $\phi_1, \dots, \phi_k \in \text{Rat}(X)^\times$ and $r_1, \dots, r_k \in \mathbb{R}$ such that

$$\overline{D}' = \overline{D} + r_1 \widehat{(\phi_1)} + \dots + r_k \widehat{(\phi_k)}$$

and $\iota(Y)$ is not contained in $\text{Supp}(\widehat{(\phi_i)})$ for every i . So $\iota^* \overline{D}' \sim_{\mathbb{R}} \iota^* \overline{D}$.

The functoriality of the pairs can now be described as follows. Let $\mu : X' \rightarrow X$ be a birational morphism of normal projective varieties. We can consider a *pull-back* of $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ defined by

$$(2.22) \quad \mu_*^{-1} : \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X'), \quad (\overline{D}; \mathcal{V}) \mapsto (\mu^* \overline{D}; \mathcal{V}^\mu),$$

where we set

$$(2.23) \quad \mathcal{V}^\mu := \sum_{\nu \in \mathfrak{V}(\text{Rat}(X'))} \nu(\mathcal{V}) [\nu \circ \mu^{*-1} : \text{Rat}(X')^\times \rightarrow \mathbb{Z}].$$

Note that μ is isomorphic over an open subscheme U of X with $\text{codim}(X \setminus U, X) \geq 2$. If Z is a prime Weil divisor on X , then

$$(2.24) \quad \text{ord}_Z = \text{ord}_{Z'} : \text{Rat}(X)^\times \rightarrow \mathbb{Z}$$

holds for the strict transform Z' of Z via μ .

Let ν_1, \dots, ν_l be divisorial valuations of $\text{Rat}(X)$ and let a_1, \dots, a_l be real numbers. By applying Lemma 2.4(2) to ν_1, \dots, ν_l successively, one can find a birational projective morphism $\mu : X' \rightarrow X$ such that X' is smooth and prime divisors Y_1, \dots, Y_l on X' such that ν_i is equivalent to ord_{Y_i} for each i . One then has

$$(2.25) \quad \mu_*^{-1} \left(\overline{D}; \sum_{i=1}^l a_i [\nu_i] \right) = \left(\overline{D}; \sum_{i=1}^l a_i [Y_i] \right) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X').$$

If $(\overline{D}; [E]) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X)$, we can consider another *pull-back* of $(\overline{D}; [E])$ defined as

$$(2.26) \quad \mu^* : \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X'), \quad (\overline{D}; [E]) \mapsto (\mu^* \overline{D}; [\mu^* E]).$$

Lemma 2.9. *Let $\mu : X' \rightarrow X$ be a birational morphism of normal projective varieties and let $\nu \in \mathfrak{V}(\text{Rat}(X))$.*

- (1) Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$. For $\mathbb{K} = \mathbb{R}, \mathbb{Q}$, and a blank and $? = ss$ and s , one has

$$\widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V}) \stackrel{\mu^*}{=} \widehat{\Gamma}_{\mathbb{K}}^?(\mu^* \overline{D}; \mathcal{V}^\mu).$$

In particular, $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ is effective (respectively, strictly effective, ν -effective, strictly ν -effective) if and only if so is $\mu_*^{-1}(\overline{D}; \mathcal{V})$.

- (2) Let $(\overline{D}; [E]) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X)$. For $\mathbb{K} = \mathbb{R}, \mathbb{Q}$, and a blank and $? = ss$ and s , one has

$$\widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; [E]) \stackrel{\mu^*}{=} \widehat{\Gamma}_{\mathbb{K}}^?(\mu^* \overline{D}; [\mu^* E]).$$

In particular, $(\overline{D}; [E]) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X)$ is effective (respectively, strictly effective, ν -effective, strictly ν -effective) if and only if so is $\mu^*(\overline{D}; [E])$.

Proof. If $\phi \in \widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V})$, then it is clear that $\mu^* \phi \in \widehat{\Gamma}_{\mathbb{K}}^?(\mu^* \overline{D}; \mathcal{V}')$. Suppose that $\phi' \in \widehat{\Gamma}_{\mathbb{K}}^?(\mu^* \overline{D}; \mathcal{V}') \setminus \{0\}$ and set $\phi := \mu^{*-1}(\phi')$. Since X is normal, one has $\phi \in \widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V})$.

Note that a $\phi \in \widehat{\Gamma}_{\mathbb{K}}^?(\overline{D}; \mathcal{V}) \setminus \{0\}$ satisfies $\nu_X(D + (\phi)) = \nu(\mathcal{V})$ if and only if $\nu_{X'}(\mu^* D + (\mu^* \phi)) = \nu(\mathcal{V}^\mu)$ (see Remark 2.1).

Similar arguments also imply the assertion (2). \square

2.3. Arithmetic ampleness. This subsection and the next are devoted to show some fundamental properties of “arithmetically ample” adelic \mathbb{R} -Cartier divisors. As in Notation and terminology 5, we consider two notions of “arithmetic ampleness”, which we call the “weak ampleness” (see Lemma 2.10) and the “ampleness (in the sense of Zhang)” (see Theorem 2.11).

Lemma 2.10. *Let X be a normal projective K -variety and let $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$.*

- (1) *If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is w -ample, then so is $\overline{A} + (\widehat{\phi})$ for every $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$.*
- (2) *Let Y be a closed subvariety of X . If \overline{A} is a w -ample adelic \mathbb{R} -Cartier divisor on X such that $Y \not\subset \text{Supp}(\overline{A})$, then the restriction $\overline{A}|_Y$ is again w -ample.*
- (3) *Let $\nu \in \mathfrak{V}(\text{Rat}(X))$. If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is w -ample, then there exists a $\phi \in \widehat{\Gamma}_{\mathbb{R}}^{\text{ss}}(\overline{A})$ such that $\overline{A} + (\widehat{\phi}) >_{\nu} 0$.*
- (4) *Let $\overline{D}_1, \dots, \overline{D}_m \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $v_1, \dots, v_l \in M_K \cup \{\infty\}$, and $\varphi_1 \in C_{v_1}^0(X), \dots, \varphi_l \in C_{v_l}^0(X)$. If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is w -ample, then there exists an $\varepsilon > 0$ such that*

$$\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k])$$

is also w -ample for every ε_i, φ_k with $|\varepsilon_i| \leq \varepsilon$ and $\|\varphi_k\|_{\text{sup}} \leq \varepsilon$.

- (5) *For any w -ample adelic \mathbb{R} -Cartier divisor \overline{A} on X , there exists a w -ample adelic \mathbb{Q} -Cartier divisor \overline{A}' such that $\overline{A} > \overline{A}'$.*

Proof. (1): By definition (see Notation and terminology 5), we can write

$$\overline{A} = \sum_{k=1}^l a_k \overline{A}_k$$

with $l \geq 1$, $a_k > 0$, and adelic Cartier divisors \overline{A}_k on X such that, for each k , \overline{A}_k is ample and $H^0(m\overline{A}_k) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}_k) \rangle_K$ for every $m \gg 1$. We write $\phi =$

$\phi_1^{\otimes e_1} \otimes \cdots \otimes \phi_r^{\otimes e_r}$ with $\phi_1, \dots, \phi_r \in \text{Rat}(X)$ and positive numbers e_1, \dots, e_r . Then

$$\begin{aligned} \overline{A} + (\widehat{\phi}) &= \sum_{k=1}^l a_k \overline{A}_k + \sum_{j=1}^r e_j (\widehat{\phi_j}) \\ &= \sum_{j=1}^r e_j (b_j \overline{A}_1 + (\widehat{\phi_j})) + \left(a_1 - \sum_{j=1}^r e_j b_j \right) \overline{A}_1 + \sum_{k=2}^l a_k \overline{A}_k \end{aligned}$$

is w-ample for every positive rational numbers b_1, \dots, b_r with $\sum_{j=1}^r e_j b_j \leq a_1$.

(2): Assume that \overline{A} is an adelic Cartier divisor on X such that A is ample, such that $H^0(mA) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}) \rangle_K$ for every $m \gg 1$, and such that $Y \not\subset \text{Supp}(A)$. For each $m \geq 1$, we have a diagram

$$\begin{array}{ccc} \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}|_Y) \rangle_K & \longrightarrow & H^0(mA|_Y) \\ \uparrow & & \uparrow \\ \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}) \rangle_K & \longrightarrow & H^0(mA). \end{array}$$

For every $m \gg 1$, $\langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}) \rangle_K = H^0(mA)$ and $H^0(mA) \rightarrow H^0(mA|_Y)$ is surjective, so that we can obtain $\langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}|_Y) \rangle_K = H^0(mA|_Y)$ for every $m \gg 1$.

In general, a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} is a positive \mathbb{R} -linear combination $\sum_{k=1}^l a_k \overline{A}_k$ such that A_k is an ample Cartier divisor on X , and such that $H^0(mA_k) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}_k) \rangle_K$ for every $m \gg 1$. For each k , we take a $\phi_k \in \text{Rat}(X)^\times$ such that $Y \not\subset \text{Supp}(A_k + (\phi_k))$, and set $\overline{A}' := \overline{A} + \sum_{k=1}^l a_k (\widehat{\phi_k})$. By the above arguments,

$$\overline{A}'|_Y = \sum_{k=1}^l a_k (\overline{A}_k + (\widehat{\phi_k}))|_Y$$

is a w-ample adelic \mathbb{R} -Cartier divisor on Y . By Lemma 2.2(2), there exist $\psi_1, \dots, \psi_r \in \text{Rat}(X)^\times$ and $b_1, \dots, b_r \in \mathbb{R}$ such that $\overline{A} - \overline{A}' = \sum_{i=1}^r b_i (\widehat{\psi_i})$, and such that $Y \not\subset \text{Supp}((\psi_i))$ for every i . Hence

$$\overline{A}|_Y = \overline{A}'|_Y + \sum_{i=1}^r b_i (\widehat{\psi_i}|_Y)$$

is w-ample by the assertion (1).

(3): There exists a $\phi \in \widehat{\Gamma}_{\mathbb{R}}^{\text{ss}}(\overline{A})$ such that $c_X(\nu) \notin \text{Supp}(A + (\phi))$, so $\nu_X(A + (\phi)) = 0$.

(4): Write $\overline{A} = \sum_{k=1}^l a_k \overline{A}_k$ as above. Without loss of generality, one can assume that $\overline{D}_i \in \widehat{\text{Div}}(X)$ for every i , and that $l = 0$. By [19, Proposition 5.3(5)], one finds a positive rational number $\varepsilon' > 0$ such that, for every i , $A_1 \pm \varepsilon' D_i$ is ample and $H^0(m(A_1 \pm \varepsilon' D_i)) = \langle \widehat{\Gamma}^{\text{ss}}(m(\overline{A}_1 \pm \varepsilon' \overline{D}_i)) \rangle_K$ for every sufficiently divisible m . Then

$$\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i = \sum_{i=1}^m \frac{|\varepsilon_i|}{\varepsilon'} (\overline{A}_1 + \text{sgn}(\varepsilon_i) \varepsilon' \overline{D}_i) + \left(a_1 - \sum_{i=1}^m \frac{|\varepsilon_i|}{\varepsilon'} \right) \overline{A}_1 + \sum_{k=2}^l a_k \overline{A}_k$$

is w-ample for every real numbers ε_i with $\sum_{i=1}^m |\varepsilon_i| \leq \varepsilon' a_1$.

The assertion (5) results from definition and the assertion (4) above. \square

Theorem 2.11. *Let $\pi : X \rightarrow \text{Spec}(K)$ be a normal projective K -variety.*

- (1) If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is relatively nef and w -ample, then \overline{A} is ample.
- (2) If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is ample and $\overline{N} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is nef, then $\overline{A} + \overline{N}$ is also ample.
- (3) For an $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, the following are equivalent.
 - (a) \overline{A} is ample.
 - (b) A is ample, \overline{A} is relatively nef, and $\inf_{x \in X(\overline{K})} h_{\overline{A}}(x) > 0$.
 - (c) A is ample and there exists an $\varepsilon > 0$ such that $\overline{A} - \pi^* \overline{N}$ is nef for every $\overline{N} \in \widehat{\text{Div}}_{\mathbb{R}}(\text{Spec}(K))$ with $0 < \widehat{\deg}(\overline{N}) \leq \varepsilon$.
 - (d) A is ample and $\overline{A} - \pi^* \overline{N}$ is nef for an $\overline{N} \in \widehat{\text{Div}}_{\mathbb{R}}(\text{Spec}(K))$ with $\overline{N} > 0$.
- (4) Let \overline{A} be a relatively nef adelic \mathbb{Q} -Cartier divisor on X . Then \overline{A} is ample if and only if \overline{A} is w -ample.
- (5) Let $\overline{A}_1, \dots, \overline{A}_l$ be relatively nef and w -ample adelic \mathbb{Q} -Cartier divisors on X . Then $\overline{N} + \alpha_1 \overline{A}_1 + \dots + \alpha_l \overline{A}_l$ is w -ample for every positive real numbers $\alpha_1, \dots, \alpha_l$ and for every nef adelic \mathbb{R} -Cartier divisor \overline{N} such that N is contained in the rational \mathbb{R} -subspace spanned by A_1, \dots, A_l .

Proof. (1): Let $x \in X(\overline{K})$. There is a $\phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\overline{A} + (\widehat{\phi})$ can be written as $\sum_{k=1}^l a_k \overline{A}_k$ with positive real numbers a_1, \dots, a_l and adelic Cartier divisors $\overline{A}_1, \dots, \overline{A}_l$ such that A_k are ample, such that, for each k , $H^0(mA_k) = \widehat{\Gamma}^{\text{ss}}(m\overline{A}_k)_K$ for every $m \gg 1$, and such that $x \notin \bigcup_k \text{Supp}(A_k)$. Thus

$$h_{\overline{A}}(x) = h_{\overline{A} + (\widehat{\phi})}(x) = \sum_{k=1}^l a_k h_{\overline{A}_k}(x) > 0,$$

and \overline{A} is nef.

By Lemma 2.10(2), given any closed subvariety Y of X , $\overline{A}|_Y$ is nef and w -ample, so $\widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y + 1)}) = \widehat{\text{vol}}(\overline{A}|_Y) > 0$.

(2): For every closed subvariety Y of X , we have

$$\begin{aligned} \widehat{\deg}((\overline{A}|_Y + \overline{N}|_Y)^{\cdot(\dim Y + 1)}) \\ = \sum_{i=0}^{\dim Y + 1} \binom{\dim Y + 1}{i} \widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y + 1 - i)} \cdot (\overline{N}|_Y)^{\cdot i}). \end{aligned}$$

Since $\widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y + 1 - i)} \cdot (\overline{N}|_Y)^{\cdot i}) \geq 0$ for every i and $\widehat{\deg}((\overline{A})^{\cdot(\dim Y + 1)}) > 0$, we conclude that $\widehat{\deg}((\overline{A}|_Y + \overline{N}|_Y)^{\cdot(\dim Y + 1)}) > 0$.

(3): The implication (c) \Rightarrow (d) are obvious.

(a) \Rightarrow (b): Obviously, \overline{A} is nef. So, for every closed subvariety Y of X , we have $\widehat{\text{vol}}(\overline{A}|_Y) = \widehat{\deg}((\overline{A}|_Y)^{\cdot(\dim Y + 1)}) > 0$. This implies that $A|_Y$ is big for every closed variety Y of X . Hence, by the Nakai-Moishezon criterion, A is ample.

We are going to show $\inf_{x \in X(\overline{K})} h_{\overline{A}}(x) > 0$ by induction on dimension (see [29, Proof of Lemma 1.3]). We can assume that $\dim X$ is positive. Since $\widehat{\text{vol}}(\overline{A}) = \widehat{\deg}(\overline{A}^{\cdot(\dim X + 1)}) > 0$, \overline{A} is big, so there exists a $\phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\overline{A} + (\widehat{\phi}) > 0$.

For every $x \in X(\overline{K})$ with $x \notin \text{Supp}(A + (\phi))$,

$$\widehat{\deg}(\overline{A}|_x) = \widehat{\deg}((\overline{A} + (\widehat{\phi}))|_x) \geq \frac{[k(x) : \mathbb{Q}]}{2} \text{ess.inf}_{p \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{A} + (\widehat{\phi})}(p) > 0,$$

where $k(x)$ denotes the residue field of the image of x . Set $Z := \text{Supp}(A + (\phi))$ endowed with the reduced induced scheme structure. By the induction hypothesis,

$$\inf_{x \in X(\overline{K})} h_{\overline{A}}(x) \geq \min \left\{ \frac{[K : \mathbb{Q}]}{2} \text{ess.inf}_{p \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{A} + (\widehat{\phi})}(p), \inf_{x \in Z(\overline{K})} h_{\overline{A}}(x) \right\} > 0.$$

(b) \Rightarrow (c): Set $\varepsilon := \inf_{x \in X(\overline{K})} h_{\overline{A}}(x) > 0$, and let $\overline{N} \in \widehat{\text{Div}}_{\mathbb{R}}(\text{Spec}(K))$ such that $\widehat{\deg}(\overline{N}) \leq \varepsilon$. For every $x \in X(\overline{K})$, we have

$$h_{\overline{A} - \pi^* \overline{N}}(x) \geq h_{\overline{A}}(x) - \varepsilon \geq 0.$$

So $\overline{A} - \pi^* \overline{N}$ is nef.

(d) \Rightarrow (a): Both $\pi^* \overline{N}$ and \overline{A} are nef. For every closed subvariety Y , we have

$$\begin{aligned} & \widehat{\deg} \left((\overline{A}|_Y)^{\cdot(\dim Y + 1)} \right) \\ &= \sum_{i=0}^{\dim Y + 1} \binom{\dim Y + 1}{i} \widehat{\deg} \left((\overline{A}|_Y - \pi^* \overline{N}|_Y)^{\cdot(\dim Y + 1 - i)} \cdot (\pi^* \overline{N}|_Y)^{\cdot i} \right) \\ &\geq (\dim Y + 1) \widehat{\deg} \left((\overline{A}|_Y - \pi^* \overline{N}|_Y)^{\cdot \dim Y} \cdot (\pi^* \overline{N}|_Y) \right) \\ &= (\dim Y + 1) \deg \left((A|_Y)^{\cdot \dim Y} \cdot \widehat{\deg}(\overline{N}) \right) > 0. \end{aligned}$$

(4): The “if” part is nothing but the assertion (1), so we are going to show the “only if” part. Let U be an open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{A} exists. By the assertion (3) above, one can find an $\varepsilon > 0$ such that

$$\overline{A} - \pi^* \left(0, \sum_{v \notin U} \varepsilon[v] \right)$$

is ample. By definition, there exists an O_K -model $(\mathcal{X}_{\varepsilon}, \mathcal{A}_{\varepsilon})$ of (X, A) such that $(\mathcal{X}_{\varepsilon}, \mathcal{A}_{\varepsilon})|_U$ is a model of definition for \overline{A} , $\mathcal{A}_{\varepsilon}$ is a relatively nef \mathbb{Q} -Cartier divisor on $\mathcal{X}_{\varepsilon}$, and

$$(2.27) \quad \overline{A} \geq \left(\mathcal{A}_{\varepsilon}, g_{\infty}^{\overline{A}} \right) \geq \overline{A} - \pi^* \left(0, \sum_{v \notin U} \varepsilon[v] \right).$$

Therefore, $\left(\mathcal{A}_{\varepsilon}, g_{\infty}^{\overline{A}} \right)$ is also ample. By the arithmetic Nakai-Moishezon criterion [31, Theorems (3.5) and (4.2)], $\left(\mathcal{A}_{\varepsilon}, g_{\infty}^{\overline{A}} \right)$ is w-ample, and so is \overline{A} .

(5): There exist β_1, \dots, β_l such that $0 < \beta_i < \alpha_i$ for every i and $N + \beta_1 A_1 + \dots + \beta_l A_l$ is rational. By the assertions (1), (2), and (4) above, $\overline{N} + \beta_1 \overline{A}_1 + \dots + \beta_l \overline{A}_l$ is w-ample, and so is

$$\overline{N} + \sum_{i=1}^l \alpha_i \overline{A}_i = \left(\overline{N} + \sum_{j=1}^l \beta_j \overline{A}_j \right) + \sum_{k=1}^l (\alpha_k - \beta_k) \overline{A}_k.$$

□

Remark 2.12. In [7, Remark 3.20], Burgos Gil, Moriwaki, Philippon, and Sombra proposed a question whether an ample adelic \mathbb{R} -Cartier divisor \overline{D} on X is w-ample or not. This question is known to have positive answer in the following cases.

- (1) \overline{D} is an ample adelic \mathbb{Q} -Cartier divisor (see Theorem 2.11(4)).

- (2) X has dimension one (see Corollary A.4 below).
- (3) \overline{D} is a toric metrized \mathbb{R} -Cartier divisor on a projective toric variety X (see [7, Corollary 6.3(2)]).

2.4. Arithmetic base loci.

Definition 2.7. Let X be a normal projective variety over a field. Recall that the *augmented stable base locus* of an \mathbb{R} -Weil divisor D is defined as

$$(2.28) \quad \mathbf{B}_+(D) := \bigcap_{A: \text{ ample}} \mathbf{B}(D - A),$$

where $\mathbf{B}(D - A)$ denotes the real stable base locus of $D - A$ and the intersection is taken over all the ample \mathbb{R} -Cartier divisors A on X . (see [1, section 3.5] and [13, section 1] for detail).

Suppose that X is defined over K , and let $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$. The \mathbb{R} -linear map $[\cdot] : \text{WDiv}_{\mathbb{R}}(X) \rightarrow \text{BC}_{\mathbb{R}}(X)$ admits a natural retraction $W_X : \text{BC}_{\mathbb{R}}(X) \rightarrow \text{WDiv}_{\mathbb{R}}(X)$ defined by

$$(2.29) \quad \mathcal{V} \mapsto \sum_{\dim(\mathcal{O}_{X, c_X(\nu)})=1} \nu(\mathcal{V}) \overline{\{c_X(\nu)\}}.$$

Let $?$ = ss or s. In view of Lemma 2.3(2), we define the *real stable base locus* of a pair $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ as

$$(2.30) \quad \widehat{\mathbf{B}}^?(\overline{D}; \mathcal{V}) := \bigcap_{\phi \in \widehat{\Gamma}_{\mathbb{R}}^?(\overline{D}; \mathcal{V})} \text{Supp}(D + (\phi) - W_X(\mathcal{V}_+)),$$

and the *augmented stable base locus* of $(\overline{D}; [\Xi])$ as

$$(2.31) \quad \widehat{\mathbf{B}}_+(\overline{D}; \mathcal{V}) := \bigcap_{\overline{A}: \text{ w-ample}} \widehat{\mathbf{B}}^{\text{ss}}(\overline{D} - \overline{A}; \mathcal{V}),$$

where the intersection is taken over all the w-ample adelic \mathbb{R} -Cartier divisors \overline{A} on X (see Notation and terminology 5).

It follows from definition that all of these are Zariski closed subsets of X , that

$$(2.32) \quad \widehat{\mathbf{B}}^?(\overline{D}; \mathcal{V}) = \widehat{\mathbf{B}}^?(\overline{D}; W_X(\mathcal{V}_+)) \quad \text{and} \quad \widehat{\mathbf{B}}_+(\overline{D}; \mathcal{V}) = \widehat{\mathbf{B}}_+(\overline{D}; W_X(\mathcal{V}_+)),$$

and that

$$(2.33) \quad \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2) \subset \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}_1; \mathcal{V}_1) \cup \widehat{\mathbf{B}}^{\text{s}}(\overline{D}_2; \mathcal{V}_2)$$

holds for every $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ with effective base conditions $\mathcal{V}_1 \geq 0$ and $\mathcal{V}_2 \geq 0$.

If $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is w-ample, then $\widehat{\mathbf{B}}_+(\overline{A}) = \emptyset$ (see also Proposition 2.16(1)) and, if $\overline{E} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ is effective, then $\widehat{\mathbf{B}}^{\text{s}}(\overline{E}; [E]) = \emptyset$.

Proposition 2.13. *If $(\overline{D}; [\Xi]) \in \widehat{\text{WDiv}}_{\mathbb{Q}, \mathbb{Q}}(X)$, then*

$$\widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [\Xi]) = \bigcap_{\phi \in \widehat{\Gamma}_{\mathbb{Q}}^?(\overline{D}; [\Xi])} \text{Supp}(D + (\phi) - \Xi_+).$$

Proof. The inclusion \subset is clear. We can assume $(\overline{D}; [\Xi]) \in \widehat{\text{WDiv}}(X)$ and $\Xi \geq 0$. Suppose that $x \notin \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [\Xi])$, so that there are $\phi_1, \dots, \phi_r \in \text{Rat}(X)^\times$ and $e_1, \dots, e_r \in \mathbb{R}$ such that

$$\overline{D} + \sum_{i=1}^r e_i \widehat{(\phi_i)} > 0, \quad D + \sum_{i=1}^r e_i (\phi_i) \geq \Xi, \quad \text{and} \quad x \notin \text{Supp} \left(D + \sum_{i=1}^r e_i (\phi_i) - \Xi \right).$$

If $(e_1, \dots, e_r) \in \mathbb{Q}^r$, then we have nothing to show, so, by the same arguments as in Lemma 2.2(2), we may assume that e_1, \dots, e_r are \mathbb{Q} -linearly independent.

We denote by \overline{D}_a the adelic \mathbb{R} -Cartier divisor $\overline{D} + \sum_{i=1}^r a_i \widehat{(\phi_i)}$ for $a = (a_1, \dots, a_r) \in \mathbb{R}^r$. Let V be the rational \mathbb{R} -subspace of $\text{WDiv}_{\mathbb{R}}(X)$ generated by the components of D , Ξ , and (ϕ_i) 's. Let W be the rational \mathbb{R} -subspace of V generated by (ϕ_i) 's. Then

$$(2.34) \quad P := \{D' \in V : D' - D \in W, D' \geq \Xi, \text{ and } x \notin \text{Supp}(D' - \Xi)\}$$

is a convex rational polytope containing D_e for $e := (e_1, \dots, e_r)$ in its relative interior. By the following claim and $\text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{D}_e}(x) > 0$, one finds a rational point $f = (f_1, \dots, f_r)$ such that $D_f \in P$ and $\text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{D}_f}(x) > 0$.

Claim 2.14. *The function*

$$P \rightarrow \mathbb{R}, \quad D' \mapsto \text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{D}'}(x),$$

is continuous over the relative interior of P .

Proof. For $D', D'' \in P$ and $0 \leq \lambda \leq 1$, we have

$$\text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{(1-\lambda)\overline{D}' + \lambda\overline{D}''}(x) \leq (1-\lambda) \cdot \text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{D}'}(x) + \lambda \cdot \text{ess.inf}_{x \in X_{\infty}^{\text{an}}} g_{\infty}^{\overline{D}''}(x).$$

So, by [12, Theorem 6.3.4], the function is continuous over the relative interior of P . \square

Let $m \geq 1$ be an integer such that $mD_f \in \text{Div}(X)$. Since $\widehat{\Gamma}^f(m\overline{D}_f)$ is a full-rank lattice in $H^0(mD_f)$ (see Notation and terminology 4), there exists an integer $p \geq 1$ such that $p \cdot 1 \in \widehat{\Gamma}^f(m\overline{D}_f)$. So,

$$m\overline{D}_f + \widehat{(p)} > 0, \quad mD_f + (p) \geq m\Xi, \quad \text{and} \quad x \notin \text{Supp}(mD_f + (p) - m\Xi).$$

\square

Lemma 2.15. *Let $(\overline{D}; [\Xi]) \in \widehat{\text{WDiv}}_{\mathbb{R}, \mathbb{R}}(X)$. For any w -ample adelic \mathbb{R} -Cartier divisors $\overline{A}_1, \dots, \overline{A}_l$, there exists an $\alpha > 0$ such that*

$$\widehat{\mathbf{B}}_+(\overline{D}; [\Xi]) = \widehat{\mathbf{B}}^{\text{ss}} \left(\overline{D} - \sum_{k=1}^l \alpha_k \overline{A}_k; [\Xi] \right)$$

for every α_k with $0 < \alpha_k \leq \alpha$.

Proof. Since X is a Noetherian topological space, one finds w -ample adelic \mathbb{R} -Cartier divisors $\overline{B}_1, \dots, \overline{B}_m$ such that

$$(2.35) \quad \widehat{\mathbf{B}}_+(\overline{D}; [\Xi]) = \bigcap_{j=1}^m \widehat{\mathbf{B}}^{\text{ss}}(\overline{D} - \overline{B}_j; [\Xi]).$$

By Lemma 2.10(4), there exists an $\alpha > 0$ such that

$$\overline{B}_j - \sum_{k=1}^l \alpha_k \overline{A}_k$$

are w-ample for all j and all α_k with $0 < \alpha_k \leq \alpha$. So, by (2.33) and (2.35),

$$\begin{aligned} \widehat{\mathbf{B}}_+(\overline{D}; [\Xi]) &= \bigcap_{j=1}^m \widehat{\mathbf{B}}^{\text{ss}} \left(\overline{D} - \sum_{k=1}^l \alpha_k \overline{A}_k - \left(\overline{B}_j - \sum_{k=1}^l \alpha_k \overline{A}_k \right); [\Xi] \right) \\ &\supset \widehat{\mathbf{B}}^{\text{ss}} \left(\overline{D} - \sum_{k=1}^l \alpha_k \overline{A}_k; [\Xi] \right) \\ &\supset \widehat{\mathbf{B}}_+(\overline{D}; [\Xi]) \end{aligned}$$

for every α_k with $0 < \alpha_k \leq \alpha$. This completes the proof. \square

The following is the main purpose of this subsection.

Proposition 2.16. *Let $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$.*

- (1) *\overline{A} is w-ample if and only if $\widehat{\mathbf{B}}_+(\overline{A}) = \emptyset$.*
- (2) *If $\overline{A} \in \widehat{\text{Div}}(X)$, then \overline{A} is w-ample if and only if A is ample and $H^0(mA) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}) \rangle_K$ for every $m \gg 1$.*
- (3) *Let $\overline{D}_1, \dots, \overline{D}_m \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $v_1, \dots, v_l \in M_K \cup \{\infty\}$, and $\varphi_1 \in C_{v_1}^0(X), \dots, \varphi_l \in C_{v_l}^0(X)$. Let $E_1, \dots, E_n \in \text{Div}_{\mathbb{R}}(X)$ be effective \mathbb{R} -Cartier divisors on X . If \overline{A} is w-ample, then there exists an $\varepsilon > 0$ such that*

$$\widehat{\mathbf{B}}_+ \left(\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \sum_{j=1}^n \delta_j [E_j] \right) = \emptyset$$

for every ε_i , δ_j , and φ_k with $|\varepsilon_i| \leq \varepsilon$, $0 \leq \delta_j \leq \varepsilon$, and $\|\varphi_k\|_{\text{sup}} \leq \varepsilon$, respectively.

Proof. (1): If \overline{A} is w-ample, then $\widehat{\mathbf{B}}_+(\overline{A}) = \widehat{\mathbf{B}}^{\text{ss}}(\overline{A} - \overline{A}) = \emptyset$. So we show the converse. By [19, Lemma 5.2(1)], \overline{A} is an \mathbb{R} -linear combination of adelic Cartier divisors \overline{B} on X such that B is ample and $H^0(mB) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{B}) \rangle_K$ for every $m \gg 1$. By Lemma 2.15, one finds a w-ample adelic \mathbb{R} -Cartier divisor \overline{A}' such that $\overline{A} - \overline{A}'$ is rational and such that $\widehat{\mathbf{B}}^{\text{ss}}(\overline{A} - \overline{A}') = \emptyset$.

Write $\overline{A}' = \sum_{k=1}^l a_k \overline{A}'_k$ with positive real numbers a_1, \dots, a_l and adelic Cartier divisors $\overline{A}'_1, \dots, \overline{A}'_l$ such that, for each k , A'_k is ample and $H^0(mA'_k) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}'_k) \rangle_K$ for every $m \gg 1$. By Proposition 2.13 and [19, Lemma 5.2(2)],

$$\overline{A} = (\overline{A} - \overline{A}') + \sum_{k=1}^l a_k \overline{A}'_k = (b_1 \overline{A}'_1 + (\overline{A} - \overline{A}')) + (a_1 - b_1) \overline{A}'_1 + \sum_{k=2}^l a_k \overline{A}'_k$$

is w-ample, where b_1 is a rational number with $0 < b_1 < a_1$.

(2): The “if” part is clear by definition. Assume that $\overline{A} \in \widehat{\text{Div}}(X)$ is w-ample (see Notation and terminology 5), which infers that A is ample and $\widehat{\mathbf{B}}_+(\overline{A}) = \emptyset$. So, by Proposition 2.13, and [19, Proposition 5.3(3)], we have $H^0(mA) = \langle \widehat{\Gamma}^{\text{ss}}(m\overline{A}) \rangle_K$ for every $m \gg 1$.

(3): We endow each E_j with E_j -Green functions such that $\overline{E}_j \geq 0$. By the relation (2.33), we have

$$\begin{aligned} & \widehat{\mathbf{B}}_+ \left(\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \sum_{j=1}^n \delta_j [E_j] \right) \\ & \subset \widehat{\mathbf{B}}_+ \left(\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i - \sum_{k=1}^l \|\varphi_k\|_{\sup}(0, [v_k]) - \sum_{j=1}^n \delta_j \overline{E}_j \right) \cup \widehat{\mathbf{B}}^s \left(\sum_{j=1}^n \delta_j \overline{E}_j; \sum_{j=1}^n \delta_j [E_j] \right) \\ & = \widehat{\mathbf{B}}_+ \left(\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i - \sum_{k=1}^l \|\varphi_k\|_{\sup}(0, [v_k]) - \sum_{j=1}^n \delta_j \overline{E}_j \right). \end{aligned}$$

So the assertion results from Lemma 2.10(4). \square

Lemma 2.17. *Let $\mu : X' \rightarrow X$ be a birational morphism of normal projective K -varieties, and let $\text{Ex}(\mu)$ be the exceptional locus.*

(1) *For every $(\overline{D}; [\Xi]) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X)$, one has*

$$\widehat{\mathbf{B}}_+(\overline{D}; [\Xi]) = \widehat{\mathbf{B}}_+(\overline{D}; [\Xi]_+) = \mathbf{B}_+(D - \Xi_+) \cup \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [\Xi]).$$

(2) *For every $D \in \text{Div}_{\mathbb{R}}(X)$, one has*

$$\mathbf{B}_+(\mu^* D) = \mu^{-1} \mathbf{B}_+(D) \cup \text{Ex}(\mu).$$

(3) *For every $(\overline{D}; [E]) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}(X)$, one has*

$$\widehat{\mathbf{B}}_+(\mu^* \overline{D}; [\mu^* E]) = \mu^{-1} \widehat{\mathbf{B}}_+(\overline{D}; [E]) \cup \text{Ex}(\mu).$$

Proof. (1): We assume $\Xi \geq 0$ and show the second equality. The inclusion \supset is clear by definition. If $x \notin \mathbf{B}_+(D - \Xi) \cup \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [\Xi])$, then there are an w -ample adelic \mathbb{R} -Cartier divisor \overline{A} on X , a $\phi \in H_{\mathbb{R}}^0(D - A - \Xi) \setminus \{0\}$, and a $\psi \in \widehat{\Gamma}_{\mathbb{R}}^{\text{ss}}(\overline{D}; [\Xi]) \setminus \{0\}$ such that

$$x \notin \text{Supp}(D - A + (\phi) - \Xi) \cup \text{Supp}(D + (\psi) - \Xi).$$

Since $D - A + (\phi) \geq 0$, the Green functions $g_v^{\overline{D} - \overline{A}} - \log |\phi|_v^2$ are bounded from below for all $v \in M_K \cup \{\infty\}$, and are non-negative for all but finitely many $v \in M_K \cup \{\infty\}$. So one finds a sufficiently small rational number λ and a $p \in K^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$\overline{D} - \lambda \overline{A} + (\lambda \widehat{\phi} + (1 - \lambda) \widehat{\psi} + \widehat{p}) > 0.$$

Therefore, $x \notin \widehat{\mathbf{B}}^{\text{ss}}(\overline{D} - \lambda \overline{A}; [\Xi])$.

The assertion (2) is nothing but [3, Proposition 2.3] (which is valid over arbitrary fields).

(3): By the assertions (1) and (2),

$$\begin{aligned} \widehat{\mathbf{B}}_+(\mu^* \overline{D}; [\mu^* E]) &= \mathbf{B}_+(\mu^*(D - E)) \cup \widehat{\mathbf{B}}^{\text{ss}}(\mu^* \overline{D}; [\mu^* E]) \\ &= \mu^{-1} \mathbf{B}_+(D - E) \cup \text{Ex}(\mu) \cup \mu^{-1} \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [E]) \\ &= \mu^{-1} (\mathbf{B}_+(D - E) \cup \widehat{\mathbf{B}}^{\text{ss}}(\overline{D}; [E])) \cup \text{Ex}(\mu) \\ &= \mu^{-1} \widehat{\mathbf{B}}_+(\overline{D}; [E]) \cup \text{Ex}(\mu). \end{aligned}$$

\square

2.5. Positivity of pairs. In this subsection, we introduce several positivity notions of pairs, and prove the openness of the big cones of pairs (see Theorem 2.21).

Definition 2.8. Let X be a normal projective K -variety, let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, and let \mathbb{K} be either \mathbb{R} , \mathbb{Q} , or \mathbb{Z} . We define positivity notions for pairs as follows.

(big): We say that $(\overline{D}; \mathcal{V})$ is *big* if there exists a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} such that $(\overline{D} - \overline{A}; \mathcal{V}) > 0$ (see Notation and terminology 5 and Definition 2.5). We denote by

$$\widehat{\text{BBig}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{DBig}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{WBig}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{Big}}_{\mathbb{K}, \mathbb{K}'}(X)$$

the cone of all the big pairs in

$$\widehat{\text{BDiv}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{DDiv}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{WDiv}}_{\mathbb{K}, \mathbb{K}'}(X) \supset \widehat{\text{Div}}_{\mathbb{K}, \mathbb{K}'}(X),$$

respectively (see Definition 2.5).

(pseudo-effective): We say that $(\overline{D}; \mathcal{V})$ is *pseudo-effective* if $(\overline{D} + \overline{A}; \mathcal{V})$ is big for every big adelic \mathbb{R} -Cartier divisor \overline{A} . We write

$$(\overline{D}_1; \mathcal{V}_1) \preceq (\overline{D}_2; \mathcal{V}_2)$$

if $(\overline{D}_2 - \overline{D}_1; \mathcal{V}_2 - \mathcal{V}_1)$ is pseudo-effective.

It is clear that the above positivity notions are compatible with addition: for example, if $(\overline{D}_1; \mathcal{V}_1)$ is big and $(\overline{D}_2; \mathcal{V}_2) \geq 0$, then $(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2)$ is also big.

We define the *arithmetic volume* of $(\overline{D}; \mathcal{V})$ as

$$(2.36) \quad \widehat{\text{vol}}(\overline{D}; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{N}, \\ m \rightarrow +\infty}} \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

Definition 2.9. Let X be a normal projective K -variety, let $\nu_0 \in \mathfrak{V}(\text{Rat}(X))$, and let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$.

(ν_0 -big): A pair $(\overline{D}; \mathcal{V})$ is called ν_0 -big if there exists a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} such that $(\overline{D} - \overline{A}; \mathcal{V}) >_{\nu_0} 0$. Let \mathbb{K}, \mathbb{K}' be either \mathbb{Z}, \mathbb{Q} , or \mathbb{R} . We denote by $\widehat{\text{BBig}}_{\mathbb{K}, \mathbb{K}'}(X | \nu_0)$, etc., the cone of all the ν_0 -big pairs in $\widehat{\text{BDiv}}_{\mathbb{K}, \mathbb{K}'}(X)$, etc..

(ν_0 -pseudo-effective): We say that $(\overline{D}; \mathcal{V})$ is ν_0 -pseudo-effective if $(\overline{D} + \overline{A}; \mathcal{V})$ is ν_0 -big for every ν_0 -big adelic \mathbb{R} -Cartier divisor \overline{A} , and write

$$(\overline{D}_1; \mathcal{V}_1) \preceq_{\nu_0} (\overline{D}_2; \mathcal{V}_2)$$

if $(\overline{D}_2 - \overline{D}_1; \mathcal{V}_2 - \mathcal{V}_1)$ is ν_0 -pseudo-effective.

Remark 2.18. (1) The cones $\widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X | \nu)$, etc., are not open in $\widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, etc. (see also Lemma 2.19(1) and Theorem 2.21(2)). For instance, even if \overline{D} is ν -big, $(\overline{D}; -r[\nu])$ is not ν -big for every $r > 0$.

(2) If $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, then $\widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{D}; \mathcal{V}) \neq \{0\}$. In fact, by Lemma 2.10(5), there exists a w-ample adelic \mathbb{Q} -Cartier divisor \overline{A}' such that $(\overline{D}; \mathcal{V}) > \overline{A}'$, so $\widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{D}; \mathcal{V}) \supset \widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{A}') \neq \{0\}$.

Lemma 2.19. Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, and let $\nu \in \mathfrak{V}(\text{Rat}(X))$.

- (1) The following are equivalent.
 - (a) $(\overline{D}; \mathcal{V})$ is ν -big.
 - (b) $\nu(\mathcal{V}) \geq 0$ and $(\overline{D}; \mathcal{V}_+)$ is ν -big.

Moreover, if $\mathcal{V} \in \text{WDiv}_{\mathbb{R}}(X)$ and $\dim \mathcal{O}_{X, c_X(\nu)} = 1$, then the following is also equivalent.

- (c) $\nu(\mathcal{V}) \geq 0$ and $c_X(\nu) \notin \widehat{\mathbf{B}}_+(\overline{D}; \mathcal{V})$.
- (2) The following are equivalent.
 - (a) $(\overline{D}; \mathcal{V})$ is pseudo-effective.
 - (b) If $(\overline{D}; \mathcal{V}')$ is big, then so is $(\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}')$.
- (3) The following are equivalent.
 - (a) $(\overline{D}; \mathcal{V})$ is ν -pseudo-effective.
 - (b) If $(\overline{D}; \mathcal{V}')$ is ν -big, then so is $(\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}')$.

Proof. (1): The implication (a) \Rightarrow (b) results from a remark after (2.17), and the converse results from $(0; -\mathcal{V}_-) \geq_{\nu} 0$. The equivalence (b) \Leftrightarrow (c) is obvious by definition of $\widehat{\mathbf{B}}_+(\overline{D}; \mathcal{V})$.

(3): The implication (b) \Rightarrow (a) is clear. Given a ν -big pair $(\overline{D}'; \mathcal{V}')$, one finds a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} on X such that $(\overline{D}' - 2\overline{A}; \mathcal{V}') >_{\nu} 0$. Since $(\overline{D}; \mathcal{V})$ is ν -pseudo-effective, $(\overline{D} + \overline{A}; \mathcal{V})$ is ν -big, so there is a $\phi \in \widehat{\Gamma}_{\mathbb{R}}^{\text{ss}}(\overline{D} + \overline{A}; \mathcal{V}) \setminus \{0\}$ such that $(\overline{D} + \overline{A} + \widehat{(\phi)}; \mathcal{V}) >_{\nu} 0$ (see Lemma 2.10(3)). Hence $(\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}') >_{\nu} \overline{A} - \widehat{(\phi)}$.

By the same arguments, one can show the assertion (2). \square

Lemma 2.20. *Let $\mu : X' \rightarrow X$ be a birational morphism of normal projective varieties.*

- (1) $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ is big (respectively, pseudo-effective) if and only if so is $\mu_*^{-1}(\overline{D}; \mathcal{V})$.
- (2) If $\nu \in \mathfrak{V}(\text{Rat}(X))$ and $\dim(\mathcal{O}_{X, c_X(\nu)}) = 1$, then $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ is ν -big (ν -pseudo-effective) if and only if so is $\mu_*^{-1}(\overline{D}; \mathcal{V})$.

Proof. We show the assertion (2).

(2): Let \overline{A} be a w-ample adelic \mathbb{R} -Cartier divisor on X such that $(\overline{D}; \mathcal{V}) >_{\nu} \overline{A}$. Then $(\mu^*\overline{D}; \mathcal{V}^{\mu}) >_{\nu} \mu^*\overline{A}$ and $\mu^*\overline{A}$ is ν -big, since $c_X(\nu) \notin \text{Ex}(\mu) = \widehat{\mathbf{B}}_+(\mu^*\overline{A})$ (see Lemmas 2.17(3) and 2.19(1)), so $(\mu^*\overline{D}; \mathcal{V}^{\mu})$ is ν -big.

Conversely, if \overline{A}' is a w-ample adelic \mathbb{R} -Cartier divisor on X' such that $(\mu^*\overline{D}; \mathcal{V}^{\mu}) >_{\nu} \overline{A}'$, then there exists a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} on X such that $\overline{A}' >_{\nu} \mu^*\overline{A}$ (see Lemma 2.10(3)). So $(\mu^*(\overline{D} - \overline{A}); \mathcal{V}^{\mu}) >_{\nu} 0$ and $(\overline{D}; \mathcal{V})$ is ν -big (see Lemma 2.9(1)).

Next, if $(\overline{D}; \mathcal{V})$ is ν -pseudo-effective and \overline{A} is ν -big, then $(\overline{D} + \varepsilon\overline{A}; \mathcal{V})$ is ν -big for every $\varepsilon > 0$. So $\mu^*\overline{A}$ is ν -big and $(\mu^*\overline{D} + \varepsilon\mu^*\overline{A}; \mathcal{V}^{\mu})$ is ν -big for every $\varepsilon > 0$.

Conversely, if \overline{A} is a w-ample adelic \mathbb{R} -Cartier divisor on X , then $\mu^*\overline{A}$ is ν -big. So, $\mu_*^{-1}(\overline{D} + \varepsilon\overline{A}; \mathcal{V})$ is ν -big for every $\varepsilon > 0$ and, so is $(\overline{D} + \varepsilon\overline{A}; \mathcal{V})$. \square

The main purpose of this subsection is the following.

Theorem 2.21. *Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$. Let $\overline{D}_1, \dots, \overline{D}_m \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $\mathcal{V}_1, \dots, \mathcal{V}_n \in \text{BC}_{\mathbb{R}}(X)$, $v_1, \dots, v_l \in M_K \cup \{\infty\}$, and $\varphi_1 \in C_{v_1}^0(X), \dots, \varphi_l \in C_{v_l}^0(X)$.*

- (1) *If $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, then there exists an $\varepsilon > 0$ such that*

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$$

for every ε_i, δ_j , and φ_k with $|\varepsilon_i| \leq \varepsilon$, $|\delta_j| \leq \varepsilon$, and $\|\varphi_k\|_{\sup} \leq \varepsilon$, respectively.

In particular, $\widehat{\text{B}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X)$, etc., are open cones in $\widehat{\text{B}\mathbb{D}\text{iv}}_{\mathbb{R}, \mathbb{R}}(X)$, etc..

(2) Let $\nu_0 \in \mathfrak{V}(\text{Rat}(X))$ such that

$$c_X(\nu_0) \notin \bigcup_{j=1}^n \text{Supp}_X(\mathcal{V}_j).$$

(a) If $(\overline{D}; \mathcal{V}) \in \widehat{\text{B}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$, then there exists an $\varepsilon > 0$ such that

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \in \widehat{\text{B}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every ε_i, δ_j , and φ_k with $|\varepsilon_i| \leq \varepsilon$, $|\delta_j| \leq \varepsilon$, and $\|\varphi_k\|_{\sup} \leq \varepsilon$, respectively.

(b) Suppose that $c_X(\nu_0) \notin \text{Supp}_X(\mathcal{V} - \nu_0(\mathcal{V})[\nu_0])$. If $(\overline{D}; \mathcal{V}) \in \widehat{\text{B}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$, then there exists an $\varepsilon > 0$ such that

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \delta_0[\nu_0] + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \in \widehat{\text{B}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every ε_i, δ_j , and φ_k such that $|\varepsilon_i| \leq \varepsilon$, $|\delta_j| \leq \varepsilon$ for $j = 0, 1, \dots, n$, $\delta_0 \geq -\nu_0(\mathcal{V})$, and $\|\varphi_k\|_{\sup} \leq \varepsilon$, respectively.

In particular, if $\dim \mathcal{O}_{X, c_X(\nu_0)} = 1$, then $\widehat{\text{W}\mathbb{B}\text{ig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$ is an open cone in

$$\left\{ (\overline{D}; [\Xi]) \in \widehat{\text{W}\mathbb{D}\text{iv}}_{\mathbb{R}, \mathbb{R}}(X) : \nu_0([\Xi]) \geq 0 \right\}.$$

Proof. We show the assertion (2) only. Similar arguments also implies the assertion (1).

(2)(a): We can assume that $\mathcal{V}_1, \dots, \mathcal{V}_n$ are all effective. By Lemma 2.8(3), there exists an adelic Cartier divisor \overline{E}_j such that

$$(\overline{E}_j; \mathcal{V}_j) \geq_{\nu_0} 0.$$

Therefore,

$$\begin{aligned} & \left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \\ & \geq_{\nu_0} \left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i - \sum_{\delta_j \geq 0} \delta_j \overline{E}_j - \sum_{k=1}^l \|\varphi_k\|_{\sup} (0, [v_k]); \mathcal{V} \right), \end{aligned}$$

so we can assume $l = n = 0$. We choose a w-ample adelic \mathbb{R} -Cartier divisor \overline{A} such that $(\overline{D}; \mathcal{V}) >_{\nu_0} \overline{A}$. By Lemma 2.10(4), there is an $\varepsilon > 0$ such that $\overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i$ is w-ample for every ε_i with $|\varepsilon_i| \leq \varepsilon$. Since

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i; \mathcal{V} \right) >_{\nu_0} \overline{A} + \sum_{i=1}^m \varepsilon_i \overline{D}_i$$

and the right-hand side is w-ample, we have the required assertion.

(b): By Lemma 2.8(3) and the same arguments as above, we have

Claim 2.22. *There exists a $\gamma > 0$ such that*

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \delta_0[\nu_0] + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every ε_i , δ_j , and φ_k with $|\varepsilon_i| \leq \gamma$, $0 \leq \delta_0 \leq \gamma$, $|\delta_j| \leq \gamma$ for $j = 1, \dots, n$, and $\|\varphi_k\|_{\text{sup}} \leq \gamma$.

Next, we show

Claim 2.23. *Suppose that $\nu_0(\mathcal{V}) > 0$. There exists a γ such that $0 < \gamma < \nu_0(\mathcal{V})$ and*

$$(\overline{D}; \mathcal{V} - \delta_0[\nu_0]) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every δ_0 with $0 \leq \delta_0 \leq \gamma$.

Proof of Claim 2.23. Put $\mathcal{V}^\circ := \mathcal{V} - \nu_0(\mathcal{V})[\nu_0]$. By Claim 2.22, there exists a $\gamma_0 > 0$ such that

$$(\overline{D} + \delta \overline{D}; \mathcal{V} + \delta \mathcal{V}^\circ) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every δ with $0 \leq \delta \leq \gamma_0$. So

$$\left(\overline{D}; \mathcal{V} - \frac{\delta \nu_0(\mathcal{V})}{1 + \delta} [\nu_0] \right) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every δ with $0 \leq \delta \leq \gamma_0$. Thus the assertion holds for $\gamma := \frac{\gamma_0}{1 + \gamma_0} \nu_0(\mathcal{V})$. \square

If $\nu_0(\mathcal{V}) = 0$, then the assertion is nothing but Claim 2.22, so we assume $\nu_0(\mathcal{V}) > 0$. By Claims 2.22 and 2.23, there exists a γ with $0 < \gamma < \nu_0(\mathcal{V})$ such that

$$\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \delta_0[\nu_0] + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

and

$$(\overline{D}; \mathcal{V} - \delta_0[\nu_0]) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X|\nu_0)$$

for every ε_i , δ_j , and φ_k with $|\varepsilon_i| \leq \gamma$, $0 \leq \delta_0 \leq \gamma$, $|\delta_j| \leq \gamma$ for $j = 1, \dots, n$, and $\|\varphi_k\|_{\text{sup}} \leq \gamma$. So the assertion holds for $\varepsilon := \gamma/2$. \square

2.6. Theory of Okounkov bodies. Let X be a normal, geometrically irreducible, and projective K -variety, and let D be an \mathbb{R} -Cartier divisor on X . A graded linear series

$$V_\bullet \subset \bigoplus_{m \geq 0} H^0(mD)$$

is said to *contain an ample series* if $V_m \neq \{0\}$ for every $m \gg 1$ and there exists an ample \mathbb{R} -Cartier divisor A such that

- $A \leq D$ and
- $H^0(mA) \subset V_m$ for every sufficiently divisible m

(see [4, Definition 1.1] and [23, page 1388]). Note that, here, one can change A with an ample \mathbb{Q} -Cartier divisor $A' \leq A$ having the same properties (see Lemma 2.10(5)).

Let $\nu : \text{Rat}(X)^\times \rightarrow \Lambda_\nu$ be a valuation of $\text{Rat}(X)$ with rational rank $\dim X$ (see section 2.2). Denote by $\text{pr}_1 : \mathbb{Z} \times \Lambda_\nu \rightarrow \mathbb{Z}$ and $\text{pr}_2 : \mathbb{Z} \times \Lambda_\nu \rightarrow \Lambda_\nu$ the first and the second projection, respectively. We endow the \mathbb{R} -vector space $\Lambda_\nu \otimes_{\mathbb{Z}} \mathbb{R}$ with the Lebesgue measure vol normalized by the lattice Λ_ν .

Definition 2.10. Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$ such that $\mathcal{V} \geq 0$. For $\mathbb{K} = \text{a blank}, \mathbb{Q}$, or \mathbb{R} , we define

$$(2.37) \quad H_{\mathbb{K}}^0(D; \mathcal{V}) := \left\{ \phi \in H_{\mathbb{K}}^0(D) \setminus \{0\} : \nu_X(D + (\phi)) \geq \nu(\mathcal{V}), \forall \nu \in \mathfrak{V}(\text{Rat}(X)) \right\} \cup \{0\}$$

as a subset of $H_{\mathbb{K}}^0(D)$. Set

$$(2.38) \quad S_{\nu}(D; \mathcal{V}) := \left\{ (m, \nu(\phi)) : \phi \in H^0(mD; m\mathcal{V}), m \geq 1 \right\},$$

$$(2.39) \quad S_{\nu}(D; \mathcal{V})_m := \text{pr}_2(S_{\nu}(D; \mathcal{V}) \cap \text{pr}_1^{-1}(m))$$

for each $m \geq 1$, and

$$(2.40) \quad \Delta_{\nu}(D; \mathcal{V}) := \overline{\bigcup_{m \geq 1} \frac{1}{m} S_{\nu}(D; \mathcal{V})_m}.$$

For each $m \geq 1$, we define an \mathbb{R} -filtration on $H^0(mD; m\mathcal{V}) \subset H^0(mD)$ by

$$(2.41) \quad F^{mt}(m\overline{D}; m\mathcal{V}) := \langle \widehat{\Gamma}^{\text{ss}}(m(\overline{D} - (0, 2t[\infty)])); m\mathcal{V} \rangle_K$$

for $t \in \mathbb{R}$ and set

$$(2.42) \quad R^t(\overline{D}; \mathcal{V}) := \bigoplus_{m \geq 0} F^{mt}(m\overline{D}; m\mathcal{V}).$$

Moreover, we set

$$(2.43) \quad S_{\nu}^t(\overline{D}; \mathcal{V}) := \left\{ (m, \nu(\phi)) : \phi \in F^{mt}(m\overline{D}; m\mathcal{V}) \setminus \{0\}, m \geq 1 \right\},$$

$$(2.44) \quad S_{\nu}^t(\overline{D}; \mathcal{V})_m := \text{pr}_2(S_{\nu}^t(\overline{D}; \mathcal{V}) \cap \text{pr}_1^{-1}(m))$$

for each $m \geq 1$, and

$$(2.45) \quad \Delta_{\nu}^t(\overline{D}; \mathcal{V}) := \overline{\bigcup_{m \geq 1} \frac{1}{m} S_{\nu}^t(\overline{D}; \mathcal{V})_m}.$$

We define the *concave transform* $G_{\nu}^{(\overline{D}; \mathcal{V})} : \Delta_{\nu}(D; \mathcal{V}) \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$(2.46) \quad G_{\nu}^{(\overline{D}; \mathcal{V})}(w) := \sup \{ t \in \mathbb{R} : w \in \Delta_{\nu}^t(\overline{D}; \mathcal{V}) \}$$

for $w \in \Delta(D; \mathcal{V})$, and define the *arithmetic Okounkov body* of $(\overline{D}; \mathcal{V})$ as

$$(2.47) \quad \widehat{\Delta}_{\nu}(\overline{D}; \mathcal{V}) := \left\{ (w, t) \in (\Lambda_{\nu} \otimes_{\mathbb{Z}} \mathbb{R}) \times \mathbb{R}_{\geq 0} : 0 \leq t \leq G_{\nu}^{(\overline{D}; \mathcal{V})}(w) \right\}.$$

The theory of Boucksom-Chen [4, section 2] was generalized to the case of normed graded linear series that contain ample series and belong to arithmetic \mathbb{R} -Cartier divisors over generically smooth, normal, and projective arithmetic varieties in [23, section 1]. By using the same arguments, we can easily generalize it to the case of adelicly normed graded linear series that contain ample series and belong to adelic \mathbb{R} -Cartier divisors over normal projective algebraic varieties. (Here, the adelic norms are induced from the supremum norms).

We apply this theory to our graded linear series

$$\bigoplus_{m \geq 0} H^0(mD; m\mathcal{V}) \subset \bigoplus_{m \geq 0} H^0(mD)$$

endowed with the subspace adelic norms induced from \overline{D} , and obtain the following (see [4, Theorem 2.8]).

Theorem 2.24. (1) For a $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, one has

$$\begin{aligned} \widehat{\text{vol}}(\overline{D}; \mathcal{V}) &= \lim_{m \rightarrow +\infty} \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!} \\ &= (\dim X + 1)! [K : \mathbb{Q}] \cdot \text{vol}(\widehat{\Delta}_{\nu}(\overline{D}; \mathcal{V})). \end{aligned}$$

(2) For $a \in \mathbb{R}_{>0}$ and $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, one has

$$\widehat{\text{vol}}(a\overline{D}; a\mathcal{V}) = a^{\dim X + 1} \widehat{\text{vol}}(\overline{D}; \mathcal{V}).$$

(3) If $(\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}') \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, then

$$\begin{aligned} \widehat{\text{vol}}(\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}')^{1/(\dim X + 1)} \\ \geq \widehat{\text{vol}}(\overline{D}; \mathcal{V})^{1/(\dim X + 1)} + \widehat{\text{vol}}(\overline{D}'; \mathcal{V}')^{1/(\dim X + 1)}. \end{aligned}$$

Proof. We only show the assertion (2). If a is a positive integer, then the assertion results from the assertion (1), so it follows that the assertion holds for every $a \in \mathbb{Q}_{>0}$. In general, we take two sequences of positive rational numbers $(b_i)_{i=1}^{\infty}$ and $(c_i)_{i=1}^{\infty}$ such that $b_i \leq a \leq c_i$ and $c_i - b_i \rightarrow 0$ as $i \rightarrow \infty$. Then

$$\begin{aligned} b_i^{\dim X + 1} \widehat{\text{vol}}(\overline{D}; \mathcal{V}) &= \widehat{\text{vol}}(b_i\overline{D}; b_i\mathcal{V}) \\ &\leq \widehat{\text{vol}}(a\overline{D}; a\mathcal{V}) \leq \widehat{\text{vol}}(c_i\overline{D}; c_i\mathcal{V}) = c_i^{\dim X + 1} \widehat{\text{vol}}(\overline{D}; \mathcal{V}). \end{aligned}$$

By taking $i \rightarrow \infty$, we have the assertion. \square

Theorems 2.24 and 2.21 combined with the standard argument (see [12, Theorem 6.3.4]) imply the following.

Corollary 2.25. The function

$$\widehat{\text{vol}} : \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \mathbb{R}$$

is continuous, that is, if $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, $\overline{D}_1, \dots, \overline{D}_m \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $\mathcal{V}_1, \dots, \mathcal{V}_n \in \text{BC}_{\mathbb{R}}(X)$, $v_1, \dots, v_l \in M_K \cup \{\infty\}$, and $\varphi_1 \in C_{v_1}^0(X), \dots, \varphi_l \in C_{v_l}^0(X)$, then

$$\lim_{\varepsilon_i, \delta_j, \|\varphi_k\|_{\sup} \rightarrow 0} \widehat{\text{vol}} \left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{D}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \sum_{j=1}^n \delta_j \mathcal{V}_j \right) = \widehat{\text{vol}}(\overline{D}; \mathcal{V}).$$

(Here the base conditions may not be effective.)

Remark 2.26. In fact, one can show that the function $\widehat{\text{vol}} : \widehat{\text{DDiv}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \mathbb{R}$ is globally continuous, so a pair $(\overline{D}; \mathcal{V}) \in \widehat{\text{DDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ is big if and only if $\widehat{\text{vol}}(\overline{D}; \mathcal{V}) > 0$. (Detail will appear in [20].) We will use this fact to show Theorem 4.9 below. One of the keys of the proof is that one can generalize the estimate of the volume discrepancy (see [22, Corollary 1.2.2]) to the case of the linear series.

3. APPROXIMATION OF PAIRS

3.1. Construction of adelic metrics. Let X be a normal projective K -variety, and let

$$\overline{D}_1 = \left(D_1, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}_1} \right), \dots, \overline{D}_r = \left(D_r, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}_r} \right)$$

be adelic Cartier divisors on X such that D_1, \dots, D_r are all effective. This subsection is devoted to constructing (after blowing up) a coefficient-wise minimum of $\overline{D}_1, \dots, \overline{D}_r$ (see Definition 3.1). Several special cases are already treated by many authors [25, 9, 30, 18, 19]. Put

$$(3.1) \quad \mathcal{J} := \sum_{i=1}^r \mathcal{O}_X(-D_i).$$

Let $\varphi : Y \rightarrow X$ be a birational and projective morphism such that Y is normal and $\mathcal{J}\mathcal{O}_Y$ is Cartier. Let M be an effective Cartier divisor on Y such that $\mathcal{O}_Y(-M) = \mathcal{J}\mathcal{O}_Y$, and let

$$(3.2) \quad g_v^{\overline{M}}(x) := \min_{1 \leq i \leq r} \left\{ g_v^{\overline{D}_i}(\varphi_v^{\text{an}}(x)) \right\}$$

for $v \in M_K \cup \{\infty\}$ and $x \in Y_v^{\text{an}}$.

Lemma 3.1. *For every $v \in M_K \cup \{\infty\}$, $g_v^{\overline{M}}$ is a M -Green function on Y_v^{an} .*

Proof. This follows from the same arguments as in [19, Proposition 4.7]. \square

Choose a non-empty open subset U of $\text{Spec}(O_K)$ over which models of definition for $\overline{D}_1, \dots, \overline{D}_r$ exist. Given any $\varepsilon > 0$, there exist O_K -models $(\mathcal{X}_\varepsilon, \mathcal{D}_{i,\varepsilon})$ of (X, D_i) such that \mathcal{X}_ε is normal, $\mathcal{D}_{i,\varepsilon}$ are \mathbb{Q} -Cartier divisors on \mathcal{X}_ε , $(\mathcal{X}_\varepsilon, \mathcal{D}_{i,\varepsilon})|_U$ gives a U -model of definition for \overline{D}_i , and

$$(3.3) \quad \|g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_{i,\varepsilon})} - g_v^{\overline{D}_i}\|_{\text{sup}} \leq \varepsilon$$

for every $v \in \text{Spec}(O_K) \setminus U$. Let $\mathcal{F} \in \text{Div}_{\mathbb{Q}}(\mathcal{X}_\varepsilon)$ be a suitable positive combination of vertical fibers on \mathcal{X}_ε such that

$$\mathcal{D}_{1,\varepsilon} + \mathcal{F}, \dots, \mathcal{D}_{r,\varepsilon} + \mathcal{F}$$

are all effective. Let $n \geq 1$ be an integer such that

$$n(\mathcal{D}_{1,\varepsilon} + \mathcal{F}), \dots, n(\mathcal{D}_{r,\varepsilon} + \mathcal{F})$$

belong to $\text{Div}(\mathcal{X}_\varepsilon)$, and put

$$(3.4) \quad \mathcal{J}_{\varepsilon,n} := \sum_{i=1}^r \mathcal{O}_{\mathcal{X}_\varepsilon}(-n(\mathcal{D}_{i,\varepsilon} + \mathcal{F})).$$

The restriction of $\mathcal{J}_{\varepsilon,n}$ to the generic fiber is given by $\mathcal{J}_{\varepsilon,n}|_X = \sum_{i=1}^r \mathcal{O}_X(-nD_i)$.

Lemma 3.2. *For every $k \geq 1$, we have*

$$\left(\sum_{i=1}^r \mathcal{O}_X(-kD_i) \right) \mathcal{O}_Y = \left(\sum_{i=1}^r \mathcal{O}_X(-D_i) \right)^k \mathcal{O}_Y.$$

Proof. We can assume that X is affine and that each of D_i is principal with defining equation f_i .

Claim 3.3. *Set $I := (f_1, \dots, f_r)$ and $J_k := (f_1^k, \dots, f_r^k)$. As ideals, we have*

$$I^{kr} = J_k \cdot I^{k(r-1)}.$$

Proof of Claim 3.3. Given any non-negative integers a_1, \dots, a_r with $a_1 + \dots + a_r = kr$, there exists at least one j with $a_j \geq k$. So, $f_1^{a_1} \dots f_j^{a_j-k} \dots f_r^{a_r} \in I^{k(r-1)}$. \square

By [17, Corollary 1.2.5], the above claim implies an equality

$$(3.5) \quad \overline{J_k} = \overline{(I^k)}$$

between the integral closures of ideals. (We refer to [17, section 1] for the theory of integral closures.) So by [17, Propositions 5.2.4 and 8.1.7] the normalization of the blowup $\text{Bl}_{I^k}(X)$ is the same as the normalization of the blowup $\text{Bl}_{J_k}(X)$. By persistence of integral closures [17, Remark 1.1.3(7)] and [17, Proposition 1.5.2], we have

$$I^k \mathcal{O}_Y \subset \overline{(I^k)} \mathcal{O}_Y \subset \overline{I^k} \mathcal{O}_Y = I^k \mathcal{O}_Y$$

and, similarly, $J_k \mathcal{O}_Y = \overline{J_k} \mathcal{O}_Y$. Hence the assertion results from (3.5). \square

By Lemma 3.2, φ factorizes through the normalized blowup of X along $\mathcal{J}_{\varepsilon,n}|_X$. So there exists a birational and projective \mathcal{O}_K -morphism

$$(3.6) \quad \varphi_\varepsilon : \mathcal{Y}_\varepsilon \rightarrow \mathcal{X}_\varepsilon$$

of projective arithmetic varieties such that \mathcal{Y}_ε is a normal \mathcal{O}_K -model of Y , φ_ε extends φ , and $\mathcal{J}_{\varepsilon,n} \mathcal{O}_{\mathcal{Y}_\varepsilon}$ is Cartier. If we set \mathcal{M}_ε as a \mathbb{Q} -Cartier divisor on \mathcal{Y}_ε such that

$$(3.7) \quad \mathcal{O}_{\mathcal{Y}_\varepsilon}(-n(\mathcal{M}_\varepsilon + \varphi_\varepsilon^* \mathcal{F})) = \mathcal{J}_{\varepsilon,n} \mathcal{O}_{\mathcal{Y}_\varepsilon},$$

then $(\mathcal{Y}_\varepsilon, \mathcal{M}_\varepsilon)$ is an \mathcal{O}_K -model of (Y, M) .

Lemma 3.4. (1) For $v \in M_K$ and $x \in Y_v^{\text{an}}$,

$$g_v^{(\mathcal{Y}_\varepsilon, \mathcal{M}_\varepsilon)}(x) = \min_{1 \leq i \leq r} \left\{ g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_{i,\varepsilon})}(\varphi_v^{\text{an}}(x)) \right\}.$$

(2) For $v \in M_K \cap U$, $g_v^{(\mathcal{Y}_\varepsilon, \mathcal{M}_\varepsilon)} = g_v^{\overline{M}}$.

(3) For $v \in M_K \setminus U$, $\|g_v^{(\mathcal{Y}_\varepsilon, \mathcal{M}_\varepsilon)} - g_v^{\overline{M}}\|_{\text{sup}} \leq \varepsilon$.

Proof. For each $x \in (Y \setminus \text{Supp}(M))_v^{\text{an}}$, we have

$$\begin{aligned} g_v^{(\mathcal{Y}_\varepsilon, \mathcal{M}_\varepsilon)}(x) &= -\log \max \left\{ |f|_v^2(x) : f \in \mathcal{O}_{\mathcal{Y}_\varepsilon}(-n\mathcal{M}_\varepsilon)_{r_{\mathcal{Y}_\varepsilon}(x)} \right\} \\ &= -\log \max \left\{ |h|_v^2(\varphi_v^{\text{an}}(x)) : h \in \mathcal{J}_{\varepsilon,n,r_{\mathcal{Y}_\varepsilon}(\varphi_v^{\text{an}}(x))} \right\} - g_v^{(\mathcal{X}_\varepsilon, n\mathcal{F})}(\varphi_v^{\text{an}}(x)) \\ &= \min_{1 \leq i \leq r} \left\{ g_v^{(\mathcal{X}_\varepsilon, n\mathcal{D}_{i,\varepsilon})}(\varphi_v^{\text{an}}(x)) \right\}. \end{aligned}$$

The assertions (2) and (3) result from the assertion (1) and the relation (3.3). \square

As a consequence of Lemma 3.4, we can make the following definition.

Definition 3.1. The couple

$$\left(M, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{M}}[v] \right)$$

is an adelic Cartier divisor on Y . We denote it by

$$\min_{1 \leq i \leq r}^\varphi \{\overline{D}_i\} = \left(\min_{1 \leq i \leq r}^\varphi \{D_i\}, \sum_{v \in M_K \cup \{\infty\}} \min_{1 \leq i \leq r}^\varphi \{g_v^{\overline{D}_i}\}[v] \right).$$

Remark 3.5. (1) If $\overline{D}_i \geq 0$ for every i , then $\min_i^\varphi \{\overline{D}_i\} \geq 0$.

- (2) Let $\varphi' : Y' \rightarrow X$ be another birational and projective morphism such that Y' is normal and φ' factorizes as $\varphi' = \psi \circ \varphi$. Then

$$\min_{1 \leq i \leq r}^{\varphi'} \{\overline{D}_i\} = \psi^* \min_{1 \leq i \leq r}^{\varphi} \{\overline{D}_i\}.$$

- (3) For each integer $k \geq 1$, let

$$\mathcal{J}_k := \sum_{i=1}^r \mathcal{O}_X(-kD_i)$$

and let $\varphi_k : Y_k \rightarrow X$ be the normalized blowup along \mathcal{J}_k . Then, by Lemma 3.2, φ_k factorizes through φ_1 , and

$$\min_{1 \leq i \leq r}^{\varphi_k} \{k\overline{D}_i\} = k \min_{1 \leq i \leq r}^{\varphi_k} \{\overline{D}_i\}.$$

In particular, we can define the minimum adelic \mathbb{Q} -Cartier divisor $\min_{1 \leq i \leq r}^{\varphi} \{\overline{D}_i\}$ for every $\overline{D}_1, \dots, \overline{D}_r \in \widehat{\text{Div}}_{\mathbb{Q}}(X)$ such that D_1, \dots, D_r are effective.

Lemma 3.6. *We keep the same notation as above. For every $\nu \in \mathfrak{V}(\text{Rat}(X))$, one has*

$$\nu_Y \left(\min_{1 \leq i \leq r}^{\varphi} \{D_i\} \right) = \min_{1 \leq i \leq r} \{\nu_X(D_i)\}.$$

Proof. Note that, for any effective Cartier divisor D on X with defining ideal sheaf $\mathcal{O}_X(-D)$, one has

$$(3.8) \quad \nu_X(D) = \min \{ \nu(\phi) : \phi \in \mathcal{O}_X(-D)_{c_X(\nu)} \setminus \{0\} \}.$$

In fact, we take a local equation f defining D around $c_X(\nu)$. Each $\phi \in \mathcal{O}_X(-D)_{c_X(\nu)} \setminus \{0\}$ can be written as fg with $g \in \mathcal{O}_{X, c_X(\nu)} \setminus \{0\}$. So

$$\nu(\phi) = \nu(fg) = \nu(f) + \nu(g) \geq \nu(f).$$

Let $D' := \min_{1 \leq i \leq r}^{\varphi} \{D_i\}$, let

$$\mathcal{J} := \sum_{i=1}^r \mathcal{O}_X(-D_i)$$

as in (3.1), and let f_i be a local equation defining D_i around $c_X(\nu)$. Since $\mathcal{O}_Y(-D') = \mathcal{J}\mathcal{O}_Y$, any element in $\mathcal{O}_Y(-D')_{c_Y(\nu)} \setminus \{0\}$ can be written as

$$(3.9) \quad f_1g_1 + \dots + f_rg_r$$

with $g_i \in \mathcal{O}_{Y, c_Y(\nu)}$. We remove zeros in (3.9), and assume that any partial sum of (3.9) is nonzero. Then

$$\nu(f_1g_1 + \dots + f_rg_r) \geq \min_{g_i \neq 0} \{\nu(f_i)\}.$$

So, we conclude. \square

Lemma 3.7. *We keep the same notations as above. Let \overline{D}_0 be another adelic Cartier divisor on X such that D_0 is effective and let $\mathcal{J}_0 := \sum_{i=1}^r \mathcal{O}_X(-D_i - D_0)$. Then $\mathcal{J}_0\mathcal{O}_Y$ is Cartier and*

$$\min_{1 \leq i \leq r}^{\varphi} \{\overline{D}_i + \overline{D}_0\} = \min_{1 \leq i \leq r}^{\varphi} \{\overline{D}_i\} + \varphi^* \overline{D}_0.$$

Proof. Since

$$\bigoplus_{1 \leq i \leq r} \mathcal{O}_X(-D_i - D_0) = \left(\bigoplus_{1 \leq i \leq r} \mathcal{O}_X(-D_i) \right) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D_0),$$

we have $\mathcal{I}_0 = \mathcal{I}\mathcal{O}_X(-D_0)$. It infers the assertion. \square

Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{Z}, \mathbb{R}}(X)$ such that $\widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{D}; \mathcal{V}) \neq \{0\}$. For each integer $m \geq 1$ and $\phi \in \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \setminus \{0\}$, we put

$$\lambda_{\phi} := \text{ess.inf}_{x \in X_{\infty}^{\text{an}}} \log |\phi|(x) \exp \left(\frac{1}{2} g_{\infty}^{m\overline{D}}(x) \right),$$

and consider the minimum adelic Cartier divisor of the finite family

$$(3.10) \quad \left\{ m\overline{D} + (\widehat{\phi}) - (0, 2\lambda_{\phi}[\infty]) : \phi \in \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \setminus \{0\} \right\}.$$

Let $\varphi_m : X_m \rightarrow X$ be the normalized blowup along

$$\mathcal{I}_m := \sum_{\phi \in \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \setminus \{0\}} \mathcal{O}_X(-(mD + (\phi))),$$

and set

$$(3.11) \quad \begin{aligned} \overline{M}(m\overline{D}; m\mathcal{V}) \\ := \varphi_m^*(m\overline{D}) - \min_{\phi \in \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \setminus \{0\}}^{\varphi_m} \left\{ m\overline{D} + (\widehat{\phi}) - (0, 2\lambda_{\phi}[\infty]) \right\}. \end{aligned}$$

Lemma 3.8. (1) For each $m \geq 1$ with $\widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \neq \{0\}$, the morphism

$$\langle \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \rangle_K \otimes_K \mathcal{O}_{X_m} \rightarrow \mathcal{O}_{X_m}(M(m\overline{D}; m\mathcal{V}))$$

is surjective,

$$\widehat{\Gamma}^{\text{ss}}(\overline{M}(m\overline{D}; m\mathcal{V})) \stackrel{\varphi_m^*}{=} \widehat{\Gamma}^{\text{ss}}(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m}),$$

and $\overline{M}(m\overline{D}; m\mathcal{V}) \leq (\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m})$.

(2) If $(\overline{D}; \mathcal{V})$ is big, then $(\varphi_m : X_m \rightarrow X, \overline{M}(m\overline{D}; m\mathcal{V})/m) \in \widehat{\Theta}(\overline{D}; \mathcal{V})$ for every sufficiently divisible m .

Proof. This is a version for pairs of [19, Proposition 4.7].

(1): Since the homomorphism $\langle \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V}) \rangle_K \otimes_K \mathcal{O}_X(-mD) \rightarrow \mathcal{I}_m$ is surjective, so is

$$\iota : \langle \widehat{\Gamma}^{\text{ss}}(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m}) \rangle_K \otimes_K \mathcal{O}_{X_m} \rightarrow \mathcal{O}_{X_m}(M(m\overline{D}; m\mathcal{V})).$$

For each $s \in \widehat{\Gamma}^{\text{ss}}(\overline{M}(m\overline{D}; m\mathcal{V})) \subset \widehat{\Gamma}^{\text{ss}}(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m})$, we have $\iota(s) = s$, so $\widehat{\Gamma}^{\text{ss}}(\overline{M}(m\overline{D}; m\mathcal{V}))$ is contained in the image of $\widehat{\Gamma}^{\text{ss}}(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m})$ via ι . By [19, Claim 4.9], we have

$$\widehat{\Gamma}^{\text{ss}}(\overline{M}(m\overline{D}; m\mathcal{V})) = \iota \left(\widehat{\Gamma}^{\text{ss}}(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m}) \right).$$

(2): By Lemma 3.6 and the definition (3.11), we have

$$(\varphi_m^*(m\overline{D}); (m\mathcal{V})^{\varphi_m}) \geq \overline{M}(m\overline{D}; m\mathcal{V}).$$

The rest of the assertion is obvious. \square

3.2. Arithmetic positive intersection numbers.

Definition 3.2. Let X be a normal projective K -variety. An *approximation* of a pair $(\overline{D}; \mathcal{V}) \in \widehat{\text{Big}}_{\mathbb{R}, \mathbb{R}}(X)$ is a couple $(\mu : X' \rightarrow X, \overline{M})$ of a projective and birational morphism $\mu : X' \rightarrow X$ of normal projective varieties and an $\overline{M} \in \widehat{\text{Nef}}_{\mathbb{R}}(X') \cap \widehat{\text{Big}}_{\mathbb{R}}(X')$ such that $\overline{M} \preceq \mu_*^{-1}(\overline{D}; \mathcal{V})$. We denote by $\widehat{\Theta}(\overline{D}; \mathcal{V})$ the set of all the approximations of $(\overline{D}; \mathcal{V})$, and set

$$\widehat{\Theta}_{\text{amp}}(\overline{D}; \mathcal{V}) := \left\{ (\mu, \overline{M}) : \overline{M} \in \widehat{\text{Nef}}_{\mathbb{Q}}(X) \text{ is ample and } \overline{M} \leq \mu_*^{-1}(\overline{D}; \mathcal{V}) \right\}.$$

(admissible): Let U be a non-empty open subset of $\text{Spec}(O_K)$, and let $\delta > 0$. Let $\widehat{\Theta}_{U, \delta}(\overline{D})$ be the set of all the normal O_K -models $(\mathcal{X}, \mathcal{D})$ of (X, D) such that

- $g_v^{\overline{D}} - \delta \leq g_v^{(\mathcal{X}, \mathcal{D})} \leq g_v^{\overline{D}}$ for all $v \in M_K \setminus U$ and
- $g_v^{(\mathcal{X}, \mathcal{D})} = g_v^{\overline{D}}$ for all but finitely many $v \in M_K \cap U$.

Put

$$\widehat{\Theta}_{\text{mod}}(\overline{D}) := \bigcup_{\substack{U \subset \text{Spec}(O_K), \\ \delta > 0}} \widehat{\Theta}_{U, \delta}(\overline{D}),$$

where U runs over all the non-empty open subsets of $\text{Spec}(O_K)$. Given an $(\mathcal{X}, \mathcal{D}) \in \widehat{\Theta}_{\text{mod}}(\overline{D})$, we set $\overline{\mathcal{D}} := (\mathcal{D}, g_{\infty}^{\overline{D}})$.

An *admissible approximation* of $(\overline{\mathcal{D}}; \mathcal{V})$ is a couple $(\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{M}})$ of a projective and birational O_K -morphism $\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}$ of normal projective arithmetic varieties and a nef and big arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{M}}$ on \mathcal{X}' such that \mathcal{X}'_K is smooth, such that $\tilde{\mu}^* \overline{\mathcal{D}} - \overline{\mathcal{M}}$ is an effective arithmetic \mathbb{Q} -Cartier divisor on \mathcal{X}' , and such that $\nu_{\mathcal{X}'_K}(\tilde{\mu}^* \mathcal{D}_K - \mathcal{M}_K) \geq \nu(\mathcal{V})$ for every $\nu \in \mathfrak{V}(\text{Rat}(X))$. We denote by $\widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ the set of all the admissible approximations of $(\overline{\mathcal{D}}; \mathcal{V})$.

Given two admissible approximations $(\tilde{\mu}_1 : \mathcal{X}'_1 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_1)$ and $(\tilde{\mu}_2 : \mathcal{X}'_2 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_2)$ of $(\overline{\mathcal{D}}; \mathcal{V})$, we write

$$(\tilde{\mu}_1 : \mathcal{X}'_1 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_1) \leq (\tilde{\mu}_2 : \mathcal{X}'_2 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_2)$$

if there exists a birational morphism $\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}$ of normal projective arithmetic varieties such that $\tilde{\mu}$ can be factorized into $\mathcal{X}' \xrightarrow{\tilde{\mu}'_1} \mathcal{X}'_1 \xrightarrow{\tilde{\mu}_1} \mathcal{X}$ and $\mathcal{X}' \xrightarrow{\tilde{\mu}'_2} \mathcal{X}'_2 \xrightarrow{\tilde{\mu}_2} \mathcal{X}$, respectively, and

$$\tilde{\mu}'_{1*} \overline{\mathcal{M}}_1^{\text{ad}} \leq \tilde{\mu}'_{2*} \overline{\mathcal{M}}_2^{\text{ad}}$$

holds. The set $\widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ is partially ordered with respect to this order. (See [18, section 3].)

The approximations of arithmetic \mathbb{R} -Cartier divisors is already treated in [18, section 3]. By using the approximation theorem (see for example [25, Theorem 4.1.3]), we can easily reduce our case to the case of [18, section 3] (see Proposition 3.9 below).

Proposition 3.9. *Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{Big}}_{\mathbb{R}, \mathbb{R}}(X)$, and let U be a non-empty open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{D} exists.*

- (1) *Given any $\delta > 0$, $\widehat{\Theta}_{U, \delta}(\overline{D})$ is non-empty.*
- (2) *For each $(\mathcal{X}, \mathcal{D}) \in \widehat{\Theta}_{\text{mod}}(\overline{D})$, the ordered set $\widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ is filtered.*

- (3) Let $(\varphi : X' \rightarrow X, \overline{M}) \in \widehat{\Theta}(\overline{D}; \mathcal{V})$ and let U' be a non-empty open subset of $\mathrm{Spec}(O_K)$ over which a model of definition for \overline{M} exists. Given any δ with $0 < \delta < 1$, there exists an ample adelic \mathbb{Q} -Cartier divisor \overline{H} on X' such that
- $\overline{H} - (1 - \delta)\overline{M}$ is nef and w -ample,
 - $(\overline{D} - \overline{H}; \mathcal{V})$ is big and strictly effective, and
 - \overline{H} has a U' -model of definition.
- (4) Let $(\mathcal{X}, \mathcal{D}) \in \widehat{\Theta}_{\mathrm{mod}}(\overline{D})$, let $(\varphi : X' \rightarrow X, \overline{M}) \in \widehat{\Theta}(\overline{\mathcal{D}}^{\mathrm{ad}}; \mathcal{V})$, and let U' be a non-empty open subset of $\mathrm{Spec}(O_K)$ over which a model of definition for \overline{M} exists. Given any δ with $0 < \delta < 1$, there exists a $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{H}}) \in \widehat{\Theta}_{\mathrm{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ such that
- $\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}$ is a birational and projective morphism of normal O_K -models extending φ ,
 - $\overline{\mathcal{H}}^{\mathrm{ad}}$ is ample on X' , \mathcal{H} is relatively ample on \mathcal{X}' , and
 - $(1 - \delta)\overline{M} - \delta \sum_{v \in M_K \setminus U'} (0, [v]) \preceq \overline{\mathcal{H}}^{\mathrm{ad}}$.

Proof. The assertion (1) is nothing but [25, Theorem 4.1.3].

(2): Let $(\tilde{\varphi}_1 : \mathcal{X}'_1 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_1)$ and $(\tilde{\varphi}_2 : \mathcal{X}'_2 \rightarrow \mathcal{X}, \overline{\mathcal{M}}_2)$ in $\widehat{\Theta}_{\mathrm{ad}}(\overline{\mathcal{D}}; \mathcal{V})$. By taking a modification dominating both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, one can assume $\tilde{\varphi}_1 = \tilde{\varphi}_2$ and $\mathcal{X}'_1 = \mathcal{X}'_2$. Let

$$\overline{\mathcal{F}}_i := \tilde{\varphi}_1^* \overline{\mathcal{D}} - \overline{\mathcal{M}}_i$$

for $i = 1, 2$. Let $n \geq 1$ be an integer such that both of $n\overline{\mathcal{F}}_1$ and $n\overline{\mathcal{F}}_2$ belong to $\mathrm{Div}(\mathcal{X}'_1)$, let

$$\mathcal{J}_n := \mathcal{O}_{\mathcal{X}'_1}(-n\overline{\mathcal{F}}_1) + \mathcal{O}_{\mathcal{X}'_1}(-n\overline{\mathcal{F}}_2),$$

and let $\tilde{\psi}_1 : \mathcal{X}' \rightarrow \mathcal{X}'_1$ be the normalized blowup along \mathcal{J}_n . Put

$$\overline{\mathcal{F}} := \min^{\tilde{\psi}_1} \{ \overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2 \}$$

as in [18, Proposition 3.2], $\tilde{\varphi} := \tilde{\psi}_1 \circ \tilde{\varphi}_1$, and $\overline{\mathcal{M}} := \tilde{\varphi}^* \overline{\mathcal{D}} - \overline{\mathcal{F}}$. Then, by Lemma 3.6 and [18, Lemma 3.4], $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{M}}) \in \widehat{\Theta}_{\mathrm{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ and $(\tilde{\varphi}, \overline{\mathcal{M}}) \geq (\tilde{\varphi}_i, \overline{\mathcal{M}}_i)$ for $i = 1, 2$.

(3): By use of Theorem 2.21(1), one finds a nef and w -ample adelic \mathbb{R} -Cartier divisor \overline{A} such that $(\overline{D} - \overline{A}; \mathcal{V})$ is still big, and such that \overline{A} has a U' -model of definition (see [19, Lemma 2.6]). Fix nef and w -ample adelic Cartier divisors $\overline{A}_1, \dots, \overline{A}_l$ such that A_1, \dots, A_l form a basis for a rational \mathbb{R} -subspace of $\mathrm{Div}_{\mathbb{R}}(X)$ containing both A and M , and such that $\overline{A}_1, \dots, \overline{A}_l$ have U' -models of definition. By using Theorem 2.21(1) again, one can find an $\alpha > 0$ such that

$$(3.12) \quad (\overline{D} - \overline{A}; \mathcal{V}) - \alpha_1 \overline{A}_1 - \dots - \alpha_l \overline{A}_l$$

is big for every α_k with $0 \leq \alpha_k \leq \alpha$.

Put $\overline{H}' := \delta \overline{A} + (1 - \delta)\overline{M}$, which is ample by Theorem 2.11(2), and choose real numbers β_1, \dots, β_l such that $0 \leq \beta_k \leq \alpha\delta$ and such that

$$\overline{H}'' := \overline{H}' + \beta_1 \overline{A}_1 + \dots + \beta_l \overline{A}_l$$

is rational. By Lemma 2.19(2),

$$\begin{aligned} & (\overline{D} - \overline{H}''; \mathcal{V}) \\ &= (\delta(\overline{D} - \overline{A}; \mathcal{V}) - \beta_1 \overline{A}_1 - \dots - \beta_l \overline{A}_l) + (1 - \delta)(\overline{D} - \overline{M}; \mathcal{V}) \end{aligned}$$

is big, so that we can take a $\phi \in \widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{D} - \overline{H}''; \mathcal{V}) \setminus \{0\}$ (see Remark 2.18(2)). Then $\overline{H} := \overline{H}'' - \widehat{(\phi)}$ has the required properties.

(4): By the assertion (3), we can find a $(\varphi : X' \rightarrow X, \overline{H}) \in \widehat{\Theta}_{\text{amp}}(\overline{\mathcal{D}}^{\text{ad}}; \mathcal{V})$ such that

- $(1 - \delta)\overline{M} \preceq \overline{H}$,
- $\varphi_*^{-1}(\overline{\mathcal{D}}^{\text{ad}}; \mathcal{V}) - \overline{H}$ is big, and
- \overline{H} has a U' -model of definition.

By reducing δ if necessary, we can assume that $\overline{H} - \delta \sum_{v \in M_K \setminus U'}(0, [v])$ is ample and that $\varphi_*^{-1}(\overline{\mathcal{D}}; \mathcal{V}) - \overline{H} - \delta \sum_{v \in M_K \setminus U}(0, [v])$ is big (see Theorems 2.11 and 2.21). We can find a normal O_K -model $(\mathcal{X}', \mathcal{H}')$ of (X', H) such that

- there is a birational and projective morphism $\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}$ extending φ ,
- \mathcal{H}' is relatively nef,
- $\overline{H} - \delta \sum_{v \in M_K \setminus U'}(0, [v]) \leq \overline{\mathcal{H}'}^{\text{ad}} \leq \overline{H}$, and
- $\varphi_*^{-1}(\overline{\mathcal{D}}^{\text{ad}}; \mathcal{V}) - \overline{\mathcal{H}'}^{\text{ad}}$ is big,

where $\overline{\mathcal{H}'} := (\mathcal{H}', g_{\infty}^{\overline{H}})$.

We fix ample arithmetic Cartier divisors $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$ such that $\mathcal{A}_1, \dots, \mathcal{A}_l$ are relatively ample and form a basis for a rational \mathbb{R} -subspace of $\text{Div}_{\mathbb{R}}(\mathcal{X}')$ containing both $\tilde{\varphi}^* \mathcal{D}$ and \mathcal{H}' . We choose non-negative real numbers β_1, \dots, β_l such that

$$\varphi_*^{-1}(\overline{\mathcal{D}}^{\text{ad}}; \mathcal{V}) - \overline{\mathcal{H}'}^{\text{ad}} - \beta_1 \overline{\mathcal{A}}_1^{\text{ad}} - \dots - \beta_l \overline{\mathcal{A}}_l^{\text{ad}}$$

is still big and such that $\tilde{\varphi}^* \mathcal{D} - \mathcal{H}' - \beta_1 \mathcal{A}_1 - \dots - \beta_l \mathcal{A}_l$ belongs to $\text{Div}_{\mathbb{Q}}(\mathcal{X}')$.

Take a $\phi \in \widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\tilde{\varphi}^* \overline{\mathcal{D}}^{\text{ad}} - \overline{\mathcal{H}'}^{\text{ad}} - \beta_1 \overline{\mathcal{A}}_1^{\text{ad}} - \dots - \beta_l \overline{\mathcal{A}}_l^{\text{ad}}; \mathcal{V}^{\varphi}) \setminus \{0\}$ (see Remark 2.18(2)), and set

$$\overline{\mathcal{H}} := \overline{\mathcal{H}'} + \beta_1 \overline{\mathcal{A}}_1 + \dots + \beta_l \overline{\mathcal{A}}_l - \widehat{(\phi)}.$$

Then $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{H}}) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}; \mathcal{V})$ has the required properties. \square

Definition 3.3. Let n be an integer such that $0 \leq n \leq \dim X + 1$, let $(\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_n; \mathcal{V}_n) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, and let $\overline{D}_{n+1}, \dots, \overline{D}_{\dim X + 1} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$. We define

$$(3.13) \quad \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X + 1} \\ := \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i; \mathcal{V}_i)} \widehat{\deg}(\overline{M}_1 \cdots \overline{M}_n \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X + 1}),$$

which is a positive real number.

Remark 3.10. Under the notations as above, one can easily see the following.

- (1) The map $\widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)^{\times n} \times (\widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X))^{\times (\dim X + 1 - n)} \rightarrow \mathbb{R}_{>0}$,

$$((\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_n; \mathcal{V}_n); \overline{D}_{n+1}, \dots, \overline{D}_{\dim X + 1}) \\ \mapsto \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X + 1},$$

is symmetric in $(\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_n; \mathcal{V}_n)$ (respectively, in $\overline{D}_{n+1}, \dots, \overline{D}_{\dim X + 1}$).

- (2) If $(\overline{D}'_1; \mathcal{V}'_1) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$ satisfies $(\overline{D}_1; \mathcal{V}_1) \preceq (\overline{D}'_1; \mathcal{V}'_1)$, then

$$\langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X + 1} \\ \leq \langle (\overline{D}'_1; \mathcal{V}'_1) \cdot (\overline{D}_2; \mathcal{V}_2) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X + 1}.$$

(3) For every $a \in \mathbb{R}_{>0}$,

$$\begin{aligned} & \langle (a\overline{D}_1; a\mathcal{V}_1) \cdot (\overline{D}_2; \mathcal{V}_2) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ &= a \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}. \end{aligned}$$

(4) If $(\overline{D}'_1; \mathcal{V}'_1) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, then

$$\begin{aligned} & \langle (\overline{D}_1 + \overline{D}'_1; \mathcal{V}_1 + \mathcal{V}'_1) \cdot (\overline{D}_2; \mathcal{V}_2) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \geq \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \quad + \langle (\overline{D}'_1; \mathcal{V}'_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}. \end{aligned}$$

Note that the base conditions may not be effective.

Proposition 3.11. *We use the notations as above. Let $\overline{E}_{ab} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $\mathcal{W}_{cd} \in \text{BC}_{\mathbb{R}}(X)$, $v_{ef} \in M_K \cup \{\infty\}$, and $\varphi_{ef} \in C_{v_{ef}}^0(X)$. One has*

$$\begin{aligned} & \lim_{\varepsilon_{ab}, \delta_{cd}, \|\varphi_{ef}\|_{\sup} \rightarrow 0} \left\langle \left(\overline{D}_1 + \sum_{b=1}^{p_1} \varepsilon_{1b} \overline{E}_{1b} + \sum_{f=1}^{r_1} (0, \varphi_{1f}[v_{1f}]); \mathcal{V}_1 + \sum_{d=1}^{q_1} \delta_{1d} \mathcal{W}_{1d} \right) \right. \\ & \quad \left. \cdots \left(\overline{D}_n + \sum_{b=1}^{p_n} \varepsilon_{nb} \overline{E}_{nb} + \sum_{f=1}^{r_n} (0, \varphi_{nf}[v_{nf}]); \mathcal{V}_n + \sum_{d=1}^{q_n} \delta_{nd} \mathcal{W}_{nd} \right) \right\rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ &= \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}, \end{aligned}$$

where the base conditions may not be effective.

Proof. By Remark 3.10(2), we can assume $r_1 = \cdots = r_n = 0$. By Theorem 2.21(1), there exists a $\gamma_0 > 0$ such that

$$(\overline{D}_i; \mathcal{V}_i) + \left(\sum_{b=1}^{p_i} \varepsilon_{ib} \overline{E}_{ib}; \sum_{d=1}^{q_i} \delta_{id} \mathcal{W}_{id} \right) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$$

for $i = 1, \dots, n$, ε_{ab} , and δ_{cd} with $\max\{|\varepsilon_{ab}|, |\delta_{cd}|\} \leq \gamma_0$. So,

$$(1 - \gamma)(\overline{D}_i; \mathcal{V}_i) \preceq (\overline{D}_i; \mathcal{V}_i) + \left(\sum_{b=1}^{p_i} \varepsilon_{ib} \overline{E}_{ib}; \sum_{d=1}^{q_i} \delta_{id} \mathcal{W}_{id} \right) \preceq (1 + \gamma)(\overline{D}_i; \mathcal{V}_i)$$

for every $\varepsilon_{ab}, \delta_{cd}$ with $\max\{|\varepsilon_{ab}|, |\delta_{cd}|\} \leq \gamma_0 \gamma$.

By Remark 3.10(2), (3), we have

$$\begin{aligned} & (1 - \gamma)^n \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq \left\langle \left(\overline{D}_1 + \sum_{b=1}^{p_1} \varepsilon_{1b} \overline{E}_{1b} + \sum_{f=1}^{r_1} (0, \varphi_{1f}[v_{1f}]); \mathcal{V}_1 + \sum_{d=1}^{q_1} \delta_{1d} \mathcal{W}_{1d} \right) \right. \\ & \quad \left. \cdots \left(\overline{D}_n + \sum_{b=1}^{p_n} \varepsilon_{nb} \overline{E}_{nb} + \sum_{f=1}^{r_n} (0, \varphi_{nf}[v_{nf}]); \mathcal{V}_n + \sum_{d=1}^{q_n} \delta_{nd} \mathcal{W}_{nd} \right) \right\rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq (1 + \gamma)^n \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \end{aligned}$$

for every $\varepsilon_{ab}, \delta_{cd}$ with $\max\{|\varepsilon_{ab}|, |\delta_{cd}|\} \leq \gamma_0 \gamma$. Hence the middle converges to $\langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}$ as $\max\{|\varepsilon_{ab}|, |\delta_{cd}|\} \rightarrow 0$. \square

Proposition 3.12. *Let n be an integer with $0 \leq n \leq \dim X + 1$, let $(\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_n; \mathcal{V}_n) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, and let $\overline{D}_{n+1}, \dots, \overline{D}_{\dim X+1} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$.*

(1) Let U be a non-empty open subset of $\text{Spec}(O_K)$ over which a model of definition for $\overline{D}_1, \dots, \overline{D}_n$ exist, and let $\delta > 0$. One has

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ &= \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}_{\text{amp}}(\overline{D}_i; \mathcal{V}_i)} \widehat{\deg}(\overline{M}_1 \cdots \overline{M}_n \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1}) \\ &= \sup_{(\mathcal{X}, \mathcal{D}_i) \in \widehat{\Theta}_{U, \delta}(\overline{D}_i)} \sup_{(\tilde{\mu}, \overline{\mathcal{M}}_i) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_i^{\text{ad}}; \mathcal{V}_i)} \widehat{\deg}(\overline{\mathcal{M}}_1^{\text{ad}} \cdots \overline{\mathcal{M}}_n^{\text{ad}} \cdot \tilde{\mu}^* \overline{D}_{n+1} \cdots \tilde{\mu}^* \overline{D}_{\dim X+1}). \end{aligned}$$

(2) For $\overline{D}_{n+1}^{(1)}, \overline{D}_{n+1}^{(2)} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$ one has

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot (\overline{D}_{n+1}^{(1)} + \overline{D}_{n+1}^{(2)}) \cdot \overline{D}_{n+2} \cdots \overline{D}_{\dim X+1} \\ &= \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1}^{(1)} \cdots \overline{D}_{\dim X+1} \\ & \quad + \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1}^{(2)} \cdots \overline{D}_{\dim X+1}. \end{aligned}$$

(3) Let e_1, \dots, e_r be integers such that $e_1 + \dots + e_r = n$. Then

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1)^{e_1} \cdots (\overline{D}_r; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ &= \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i; \mathcal{V}_i)} \widehat{\deg}(\overline{M}_1^{e_1} \cdots \overline{M}_r^{e_r} \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1}), \end{aligned}$$

where $\langle (\overline{D}_1; \mathcal{V}_1)^{e_1} \cdots (\overline{D}_r; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}$ is an abbreviation for

$$\overbrace{\langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_1; \mathcal{V}_1) \rangle}^{e_1} \cdots \overbrace{\langle (\overline{D}_r; \mathcal{V}_r) \cdots (\overline{D}_r; \mathcal{V}_r) \rangle}^{e_r} \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}.$$

Proof. (1): We show the second equality. The inequality \geq is clear. By Proposition 3.11, there exist $(\mathcal{X}, \mathcal{D}_1) \in \widehat{\Theta}_{U, \delta}(\overline{D}_1), \dots, (\mathcal{X}, \mathcal{D}_n) \in \widehat{\Theta}_{U, \delta}(\overline{D}_n)$ such that

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} + \varepsilon. \end{aligned}$$

Take $(\mu, \overline{M}_1) \in \widehat{\Theta}(\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1), \dots, (\mu, \overline{M}_n) \in \widehat{\Theta}(\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n)$ such that

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq \widehat{\deg}(\overline{M}_1 \cdots \overline{M}_n \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1}) + \varepsilon. \end{aligned}$$

Let U' be a non-empty open subset of $\text{Spec}(O_K)$ over which models of definition for $\overline{M}_1, \dots, \overline{M}_n$ exist. We can choose a sufficiently small $\delta > 0$ such that

$$\begin{aligned} & \widehat{\deg}(\overline{M}_1 \cdots \overline{M}_r \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1}) \\ & \leq \widehat{\deg} \left(\left((1-\delta)\overline{M}_1 - \delta \sum_{v \in M_K \setminus U'} (0, [v]) \right) \cdots \left((1-\delta)\overline{M}_r - \delta \sum_{v \in M_K \setminus U'} (0, [v]) \right) \right. \\ & \quad \left. \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1} \right) + \varepsilon. \end{aligned}$$

All in all, we can find, by Proposition 3.9, $(\tilde{\mu}, \overline{\mathcal{H}}_1) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_1; \mathcal{V}_1), \dots, (\tilde{\mu}, \overline{\mathcal{H}}_n) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_n; \mathcal{V}_n)$ such that

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \leq \widehat{\deg}(\overline{\mathcal{M}}_1 \cdots \overline{\mathcal{M}}_r \cdot \mu^* \overline{\mathcal{D}}_{n+1} \cdots \mu^* \overline{\mathcal{D}}_{\dim X+1}) + 2\varepsilon \\ & \leq \widehat{\deg}\left((\overline{\mathcal{H}}_1^{\text{ad}}) \cdots (\overline{\mathcal{H}}_n^{\text{ad}}) \cdot \tilde{\mu}^* \overline{\mathcal{D}}_{n+1} \cdots \tilde{\mu}^* \overline{\mathcal{D}}_{\dim X+1}\right) + 3\varepsilon \\ & \leq \sup_{(\tilde{\mu}, \overline{\mathcal{M}}_i) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_i^{\text{ad}}; \mathcal{V}_i)} \widehat{\deg}\left(\overline{\mathcal{M}}_1^{\text{ad}} \cdots \overline{\mathcal{M}}_n^{\text{ad}} \cdot \tilde{\mu}^* \overline{\mathcal{D}}_{n+1} \cdots \tilde{\mu}^* \overline{\mathcal{D}}_{\dim X+1}\right) + 3\varepsilon. \end{aligned}$$

So we conclude the proof.

(2): The inequality \leq is clear. Let U be a non-empty open subset of $\text{Spec}(O_K)$ over which models of definition for $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_n$ exist. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(i)} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \leq \left\langle \left(\overline{\mathcal{D}}_1 - \delta \sum_{v \in M_K \setminus U} (0, [v]); \mathcal{V}_1 \right) \cdots \left(\overline{\mathcal{D}}_n - \delta \sum_{v \in M_K \setminus U} (0, [v]); \mathcal{V}_n \right) \right\rangle \\ & \quad \cdot \overline{\mathcal{D}}_{n+1}^{(i)} \cdots \overline{\mathcal{D}}_{\dim X+1} + \varepsilon. \end{aligned}$$

for $i = 1, 2$ (see Proposition 3.11). By Proposition 3.9(1) and Remark 3.10(2), there exist $(\mathcal{X}, \mathcal{D}_1) \in \widehat{\Theta}_{U, \delta}(\overline{\mathcal{D}}_1), \dots, (\mathcal{X}, \mathcal{D}_n) \in \widehat{\Theta}_{U, \delta}(\overline{\mathcal{D}}_n)$ such that

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(i)} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \leq \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(i)} \cdots \overline{\mathcal{D}}_{\dim X+1} + \varepsilon \end{aligned}$$

for $i = 1, 2$. Hence, by Proposition 3.9(2),(4),

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(1)} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \quad + \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(2)} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \leq \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(1)} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \quad + \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot \overline{\mathcal{D}}_{n+1}^{(2)} \cdots \overline{\mathcal{D}}_{\dim X+1} + 2\varepsilon \\ & = \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n^{\text{ad}}; \mathcal{V}_n) \rangle \cdot (\overline{\mathcal{D}}_{n+1}^{(1)} + \overline{\mathcal{D}}_{n+1}^{(2)}) \cdots \overline{\mathcal{D}}_{\dim X+1} + 2\varepsilon \\ & \leq \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1) \cdots (\overline{\mathcal{D}}_n; \mathcal{V}_n) \rangle \cdot (\overline{\mathcal{D}}_{n+1}^{(1)} + \overline{\mathcal{D}}_{n+1}^{(2)}) \cdots \overline{\mathcal{D}}_{\dim X+1} + 2\varepsilon \end{aligned}$$

for every $\varepsilon > 0$.

(3): The inequality \geq is clear. Let U be a non-empty open subset of $\text{Spec}(O_K)$ over which models of definition for $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r$ exist. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} & \langle (\overline{\mathcal{D}}_1; \mathcal{V}_1)^{e_1} \cdots (\overline{\mathcal{D}}_r; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{\mathcal{D}}_{n+1} \cdots \overline{\mathcal{D}}_{\dim X+1} \\ & \leq \left\langle \left(\overline{\mathcal{D}}_1 - \delta \sum_{v \in M_K \setminus U} (0, [v]); \mathcal{V}_1 \right)^{e_1} \cdots \left(\overline{\mathcal{D}}_r - \delta \sum_{v \in M_K \setminus U} (0, [v]); \mathcal{V}_r \right)^{e_r} \right\rangle \\ & \quad \cdot \overline{\mathcal{D}}_{n+1} \cdots \overline{\mathcal{D}}_{\dim X+1} + \varepsilon. \end{aligned}$$

(see Proposition 3.11). By Proposition 3.9(1) and Remark 3.10(2), there exist $(\mathcal{X}, \mathcal{D}_1) \in \widehat{\Theta}_{U,\delta}(\overline{D}_1), \dots, (\mathcal{X}, \mathcal{D}_n) \in \widehat{\Theta}_{U,\delta}(\overline{D}_n)$ such that

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1)^{e_1} \cdots (\overline{D}_r; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1)^{e_1} \cdots (\overline{\mathcal{D}}_r^{\text{ad}}; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} + \varepsilon. \end{aligned}$$

Hence, by Proposition 3.9(2),(4),

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1)^{e_1} \cdots (\overline{D}_r; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} \\ & \leq \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1)^{e_1} \cdots (\overline{\mathcal{D}}_r^{\text{ad}}; \mathcal{V}_r)^{e_r} \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1} + \varepsilon \\ & = \sup_{(\tilde{\mu}, \tilde{\mathcal{M}}_i) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_i; \mathcal{V}_i)} \widehat{\deg} \left((\tilde{\mathcal{M}}_1^{\text{ad}})^{e_1} \cdots (\tilde{\mathcal{M}}_r^{\text{ad}})^{e_r} \cdot \tilde{\mu}^* \overline{D}_{n+1} \cdots \tilde{\mu}^* \overline{D}_{\dim X+1} \right) + \varepsilon \\ & \leq \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i; \mathcal{V}_i)} \widehat{\deg} \left(\overline{M}_1^{e_1} \cdots \overline{M}_r^{e_r} \cdot \mu^* \overline{D}_{n+1} \cdots \mu^* \overline{D}_{\dim X+1} \right) + \varepsilon \end{aligned}$$

for every $\varepsilon > 0$. \square

Definition 3.4. By Proposition 3.12(2), we can extend the map (3.13) to $\widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)^{\times n} \times \widehat{\text{Int}}_{\mathbb{R}}(X)^{\times (\dim X+1-n)} \rightarrow \mathbb{R}$,

$$\begin{aligned} & ((\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_n; \mathcal{V}_n); \overline{D}_{n+1}, \dots, \overline{D}_{\dim X+1}) \\ & \mapsto \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_n; \mathcal{V}_n) \rangle \cdot \overline{D}_{n+1} \cdots \overline{D}_{\dim X+1}. \end{aligned}$$

If $n = \dim X$, then, by using the same arguments as in [18, Proposition 3.10(3)], we have a map $\widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)^{\times \dim X} \times \widehat{\text{Div}}_{\mathbb{R}}(X) \rightarrow \mathbb{R}$,

$$\begin{aligned} & ((\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_{\dim X}; \mathcal{V}_{\dim X}); \overline{D}_{\dim X+1}) \\ & \mapsto \langle (\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_{\dim X}; \mathcal{V}_{\dim X}) \rangle \cdot \overline{D}_{\dim X+1}. \end{aligned}$$

Theorem 3.13. For every $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, one has

$$\widehat{\text{vol}}(\overline{D}; \mathcal{V}) = \langle (\overline{D}; \mathcal{V})^{\cdot (\dim X+1)} \rangle.$$

Proof. The inequality \geq results from Corollary 2.25. First, we assume $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{Z}, \mathbb{R}}(X)$. For any $\varepsilon > 0$, there exists an $m_1 \geq 1$ such that

$$(3.14) \quad \widehat{\text{vol}}(\overline{D}; \mathcal{V}) \leq \frac{\log \# \widehat{\Gamma}^{\text{ss}}(m\overline{D}; m\mathcal{V})}{m^{\dim X+1}/(\dim X+1)!} + \varepsilon$$

for every $m \geq m_1$ (see Theorem 2.24).

By Lemma 3.8(2), there exists an $m_2 \geq m_1$ such that

$$(\varphi_{m_2 m}, \overline{M}(m_2 m \overline{D}; m_2 m \mathcal{V})/(m_2 m)) \in \widehat{\Theta}(\overline{D}; \mathcal{V})$$

for every $m \geq 1$. By Lemma 3.8(1) and Theorem 2.24,

$$(3.15) \quad \frac{\log \# \widehat{\Gamma}^{\text{ss}}(mm_2 \overline{D}; mm_2 \mathcal{V})}{(mm_2)^{\dim X+1}/(\dim X+1)!} = \frac{\log \# \widehat{\Gamma}^{\text{ss}}(\overline{M}(mm_2 \overline{D}; mm_2 \mathcal{V}))}{(mm_2)^{\dim X+1}/(\dim X+1)!}$$

for every $m \geq 1$ and

$$(3.16) \quad \frac{\log \# \widehat{\Gamma}^{\text{ss}}(\overline{M}(mm_2 \overline{D}; mm_2 \mathcal{V}))}{(mm_2)^{\dim X+1}/(\dim X+1)!} \leq \frac{\widehat{\text{vol}}(\overline{M}(m_2 \overline{D}; m_2 \mathcal{V}))}{m_2^{\dim X+1}} + \varepsilon$$

for every $m \gg 1$.

All in all, we have

$$\widehat{\text{vol}}(\overline{D}; \mathcal{V}) \leq \frac{\widehat{\text{vol}}(\overline{M}(m_2 \overline{D}; m_2 \mathcal{V}))}{m_2^{\dim X + 1}} + 2\varepsilon \leq \langle (\overline{D}; \mathcal{V})^{\cdot(\dim X + 1)} \rangle + 2\varepsilon$$

for every $\varepsilon > 0$.

In general, by homogeneity and continuity (see Proposition 3.11 and Theorem 2.24), the assertion holds for every $(\overline{D}; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$. \square

4. DIFFERENTIABILITY OF THE ARITHMETIC VOLUMES

4.1. Proof of Theorem A. We recall the arithmetic Siu inequality of Yuan and its consequence (see [28, Theorem 2.2] and [18, Proposition 6.1]).

Proposition 4.1. (1) Let $\overline{M}, \overline{N}$ be nef adelic \mathbb{R} -Cartier divisors on X . Then

$$\widehat{\text{vol}}(\overline{M} - \overline{N}) \geq \widehat{\deg}(\overline{M}^{\dim X + 1}) - (\dim X + 1) \widehat{\deg}(\overline{M}^{\dim X} \cdot \overline{N}).$$

(2) Let $\overline{M}, \overline{D}' \in \widehat{\text{Div}}_{\mathbb{R}}(X)$. Suppose that \overline{M} is nef, and there exists a nef and big adelic \mathbb{R} -Cartier divisor \overline{A} such that $\overline{A} \pm \overline{D}'$ is nef and $\overline{A} - \overline{M}$ is pseudo-effective. Then for all $r \in \mathbb{R}$

$$\begin{aligned} \widehat{\text{vol}}(\overline{M} + r\overline{D}') - \widehat{\text{vol}}(\overline{M}) \\ \geq (\dim X + 1) \widehat{\deg}(\overline{M}^{\dim X} \cdot \overline{D}') \cdot r - C(|r|) \widehat{\text{vol}}(\overline{A}) \cdot r^2, \end{aligned}$$

$$\text{where } C(|r|) := 2 \dim X (\dim X + 1) (1 + |r|)^{\dim X - 1}.$$

The main purpose of this paper is the following.

Theorem 4.2. Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$ and $\overline{D}' \in \widehat{\text{Div}}_{\mathbb{R}}(X)$. If $(\overline{D}; \mathcal{V})$ is big, then the function

$$\mathbb{R} \ni r \mapsto \widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) \in \mathbb{R}$$

is two-sided differentiable at $r = 0$, and

$$\lim_{r \rightarrow 0} \frac{\widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V})}{r} = (\dim X + 1) \langle (\overline{D}; \mathcal{V})^{\cdot \dim X} \rangle \cdot \overline{D}'.$$

Proof. First, we assume that \overline{D}' is integrable, and fix a nef and big adelic \mathbb{R} -Cartier divisor \overline{A} such that $\overline{A} \pm \overline{D}'$ is nef and $\overline{A} - \overline{D}$ is pseudo-effective. By Proposition 4.1(1), for every $r \in \mathbb{R}$ with $|r| \leq 1$ and $(\varphi, \overline{M}) \in \widehat{\Theta}(\overline{D}; \mathcal{V})$,

$$\begin{aligned} \widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) &\geq \widehat{\text{vol}}(\overline{M} + r\varphi^* \overline{D}') \\ &\geq \widehat{\text{vol}}(\overline{M}) + (\dim X + 1) \widehat{\deg}(\overline{M}^{\dim X} \cdot \varphi^* \overline{D}') \cdot r - C \widehat{\text{vol}}(\overline{A}) \cdot r^2 \end{aligned}$$

and, for every $r \in \mathbb{R}$ with $|r| \leq 1$ and $(\varphi_r, \overline{M}_r) \in \widehat{\Theta}(\overline{D} + r\overline{D}'; \mathcal{V})$,

$$\begin{aligned} \widehat{\text{vol}}(\overline{D}; \mathcal{V}) &\geq \widehat{\text{vol}}(\overline{M}_r - r\varphi_r^* \overline{D}') \\ &\geq \widehat{\text{vol}}(\overline{M}_r) - (\dim X + 1) \widehat{\deg}(\overline{M}_r^{\dim X} \cdot \varphi_r^* \overline{D}') \cdot r - C \widehat{\text{vol}}(2\overline{A}) \cdot r^2, \end{aligned}$$

where we set $C := 2^{\dim X} \dim X (\dim X + 1)$. Note that $(\overline{D} + r\overline{D}'; \mathcal{V}) \in \widehat{\text{BBig}}_{\mathbb{R}}(X)$ for every r with $|r|$ sufficiently small (Theorem 2.21(1)). Hence, by Theorem 3.13,

$$\widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V}) \geq (\dim X + 1) r \langle (\overline{D}; \mathcal{V})^{\cdot \dim X} \rangle \cdot \overline{D}' - Cr^2 \widehat{\text{vol}}(\overline{A})$$

and

$$\begin{aligned} \widehat{\text{vol}}(\overline{D}; \mathcal{V}) - \widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) \\ \geq -(\dim X + 1)r \langle (\overline{D} + r\overline{D}'; \mathcal{V})^{\dim X} \rangle \cdot \overline{D}' - Cr^2 \widehat{\text{vol}}(2\overline{A}) \end{aligned}$$

hold for all r with $|r| \ll 1$. Thus, by Remark 3.10(4), we conclude.

Next, in general, one can find, by the Stone-Weierstrass theorem, a sequence of continuous functions $(f_n)_{n \geq 1}$ such that $\overline{D}' + (0, 2f_n[\infty])$ is of C^∞ -type and $\|f_n\|_{\text{sup}} \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\begin{aligned} \left| \frac{\widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V})}{r} - \frac{\widehat{\text{vol}}(\overline{D} + r(\overline{D}' + (0, 2f_n[\infty])); \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V})}{r} \right| \\ \leq (\dim X + 1)(\|f_n\|_{\text{sup}})[K : \mathbb{Q}] \text{vol}(D + rD') \end{aligned}$$

for all $r \in \mathbb{R} \setminus \{0\}$ and $n \geq 1$, we have, by Remark 3.10(4),

$$\lim_{r \rightarrow 0} \frac{\widehat{\text{vol}}(\overline{D} + r\overline{D}'; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V})}{r} = (\dim X + 1) \langle (\overline{D}; \mathcal{V})^{\dim X} \rangle \cdot \overline{D}'.$$

□

4.2. Arithmetic Bonnesen–Diskant inequalities. By the same arguments as in [29, section 2.3], we can easily see that the arithmetic Hodge index theorem is also valid for adelic \mathbb{R} -Cartier divisors.

Theorem 4.3 (Arithmetic Hodge index theorem). *Let X be a projective variety over a number field. Let \overline{D} be an integrable adelic \mathbb{R} -Cartier divisor on X , and let $\overline{H}_1, \dots, \overline{H}_{\dim X}$ be nef adelic \mathbb{R} -Cartier divisors on X such that $H_1, \dots, H_{\dim X-1}$ are big.*

- (1) *If $\deg(D \cdot H_2 \cdots H_{\dim X}) = 0$, then $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}_2 \cdots \overline{H}_{\dim X}) \leq 0$.*
- (2) *If $\widehat{\deg}(\overline{D} \cdot \overline{H}_1 \cdots \overline{H}_{\dim X}) = 0$, then $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}_2 \cdots \overline{H}_{\dim X}) \leq 0$.*

Proof. We reproduce the proof for reader's convenience.

(1): Suppose that $\overline{H}_1, \dots, \overline{H}_{\dim X-1}$ are nef adelic \mathbb{Q} -Cartier divisors on X . Choose $\overline{D}_1, \dots, \overline{D}_l \in \widehat{\text{Div}}(X)$ and $a_1, \dots, a_l \in \mathbb{R}$ such that a_1, \dots, a_l are \mathbb{Q} -linearly independent and $\overline{D} = a_1 \overline{D}_1 + \cdots + a_l \overline{D}_l$. Since

$$\sum_{i=1}^l a_i \deg(D_i \cdot H_1 \cdots H_{\dim X-1}) = 0$$

and $\deg(D_i \cdot H_1 \cdots H_{\dim X-1}) \in \mathbb{Q}$, we have $\deg(D_i \cdot H_1 \cdots H_{\dim X-1}) = 0$ for all i . So, by [29, Theorem 1.3], we have

$$\widehat{\deg}((b_1 \overline{D}_1 + \cdots + b_l \overline{D}_l)^2 \cdot \overline{H}_1 \cdots \overline{H}_{d-1}) \leq 0$$

for all $b_1, \dots, b_l \in \mathbb{Q}$, and $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}_1 \cdots \overline{H}_{d-1}) \leq 0$ by continuity.

Next, in general, we fix an ample adelic Cartier divisor \overline{A} on X . For each $i = 2, \dots, \dim X$, we choose a sequence $(\overline{A}_i^{(j)})_{j=1}^\infty$ of nef adelic \mathbb{R} -Cartier divisors, all of which are contained in a finite dimensional \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}(X)$, having the properties that $\overline{A}_i^{(j)} \rightarrow 0$ as $j \rightarrow \infty$ and $\overline{H}_i^{(j)} := \overline{H}_i + \overline{A}_i^{(j)}$ are all rational. For

each j , we set

$$\varepsilon_j := -\frac{\deg(D \cdot H_2^{(j)} \cdots H_{\dim X}^{(j)})}{\deg(A \cdot H_1^{(j)} \cdots H_{\dim X}^{(j)})} \in \mathbb{R}.$$

Then

$$\deg((D + \varepsilon_j A) \cdot H_2^{(j)} \cdots H_{\dim X}^{(j)}) = 0,$$

$\overline{H}_i^{(j)} \in \widehat{\text{Nef}}_{\mathbb{Q}}(X)$, and $H_i^{(j)}$ are big, so, by the first case, we have

$$\widehat{\deg}((\overline{D} + \varepsilon_j \overline{A}) \cdot \overline{H}_2^{(j)} \cdots \overline{H}_{\dim X}^{(j)}) \leq 0.$$

As $j \rightarrow \infty$, we have $\overline{H}_i^{(j)} \rightarrow \overline{H}_i$ and

$$\varepsilon_j \rightarrow -\frac{\deg(D \cdot H_2 \cdots H_{\dim X})}{\deg(A \cdot H_2 \cdots H_{\dim X})} = 0,$$

where $\deg(A \cdot H_2 \cdots H_{\dim X}) > 0$ since $H_2, \dots, H_{\dim X}$ are all big. So we conclude by continuity.

(2): Set $t := \deg(D \cdot H_2 \cdots H_{\dim X}) / \deg(H_1 \cdots H_{\dim X}) \in \mathbb{R}$. Since

$$\deg((D - tH_1) \cdot H_2 \cdots H_{\dim X}) = 0,$$

we have by the assertion (1)

$$\begin{aligned} & \widehat{\deg}((\overline{D} - t\overline{H}_1) \cdot \overline{H}_2 \cdots \overline{H}_{\dim X}) \\ &= \widehat{\deg}(\overline{D}^2 \cdot \overline{H}_2 \cdots \overline{H}_{\dim X}) + t^2 \widehat{\deg}(\overline{H}^2 \cdot \overline{H}_1 \cdots \overline{H}_{d-1}) \leq 0, \end{aligned}$$

so $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}_2 \cdots \overline{H}_{\dim X}) \leq 0$. \square

Corollary 4.4. *Let $\overline{D}, \overline{E}, \overline{H}_1, \dots, \overline{H}_{\dim X+1}$ be nef adelic \mathbb{R} -Cartier divisors on X .*

(1) *One has*

$$\begin{aligned} & \widehat{\deg}(\overline{D} \cdot \overline{E} \cdot \overline{H}_3 \cdots \overline{H}_{\dim X+1})^2 \\ & \geq \widehat{\deg}(\overline{D}^2 \cdot \overline{H}_3 \cdots \overline{H}_{\dim X+1}) \cdot \widehat{\deg}(\overline{E}^2 \cdot \overline{H}_3 \cdots \overline{H}_{\dim X+1}). \end{aligned}$$

(2) *For every n with $0 \leq n \leq \dim X + 1$ and for every i with $0 \leq i \leq n$, one has*

$$\begin{aligned} & \widehat{\deg}(\overline{D}^{(n-i)} \cdot \overline{E}^i \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^n \\ & \geq \widehat{\deg}(\overline{D}^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{n-i} \cdot \widehat{\deg}(\overline{E}^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^i. \end{aligned}$$

(3) *For every n with $0 \leq n \leq \dim X + 1$, one has*

$$\widehat{\deg}(\overline{H}_1 \cdots \overline{H}_{\dim X+1})^n \geq \prod_{i=1}^n \widehat{\deg}(\overline{H}_i^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1}).$$

(4) *For every n with $1 \leq n \leq \dim X$, one has*

$$\begin{aligned} & \widehat{\deg}(\overline{D}^n \cdot \overline{E}^{(\dim X - n + 1)}) \\ & \geq \widehat{\deg}(\overline{D}^{(n-1)} \cdot \overline{E}^{(\dim X - n + 2)}) \cdot \widehat{\deg}(\overline{D}^{(n+1)} \cdot \overline{E}^{(\dim X - n)}). \end{aligned}$$

(5) For every n with $1 \leq n \leq \dim X + 1$, one has

$$\begin{aligned} & \widehat{\deg}((\overline{D} + \overline{E})^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n} \\ & \geq \widehat{\deg}(\overline{D}^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n} + \widehat{\deg}(\overline{E}^n \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n}. \end{aligned}$$

Proof. These result by formally transforming the inequality in Theorem 4.3(2) as in the proof of [18, Theorem 2.9 and Remark 2.10]. \square

Proposition 4.5. Let n be an integer with $0 \leq n \leq \dim X + 1$, let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, and let $\overline{H}_{n+1}, \dots, \overline{H}_{\dim X+1} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$.

(1) One has

$$\begin{aligned} & \widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2) \\ & \geq \sum_{i=0}^{\dim X+1} \binom{\dim X+1}{i} \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X+1-i)} \rangle \end{aligned}$$

(2) For every n with $1 \leq n \leq \dim X$, one has

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot n} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X-n+1)} \rangle^2 \\ & \geq \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot (n-1)} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X-n+2)} \rangle \\ & \quad \cdot \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot (n+1)} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X-n)} \rangle. \end{aligned}$$

(3) For every n with $0 \leq n \leq \dim X + 1$, one has

$$\begin{aligned} & \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot n} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X-n+1)} \rangle^{\dim X+1} \\ & \geq \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1)^n \cdot \widehat{\text{vol}}(\overline{D}_2; \mathcal{V}_2)^{\dim X-n+1}. \end{aligned}$$

(4) For every n with $1 \leq n \leq \dim X + 1$, one has

$$\begin{aligned} & (\langle (\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2)^{\cdot n} \rangle \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n} \\ & \geq (\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot n} \rangle \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n} \\ & \quad + (\langle (\overline{D}_2; \mathcal{V}_2)^{\cdot n} \rangle \cdot \overline{H}_{n+1} \cdots \overline{H}_{\dim X+1})^{1/n}. \end{aligned}$$

Proof. (1): Given any $\varepsilon > 0$, there exist, by Proposition 3.11 and Proposition 3.9, $(\mathcal{X}, \mathcal{D}_1) \in \widehat{\Theta}_{\text{mod}}(\overline{D}_1)$ and $(\mathcal{X}, \mathcal{D}_2) \in \widehat{\Theta}_{\text{mod}}(\overline{D}_2)$ such that

$$\begin{aligned} & \sum_{i=0}^{\dim X+1} \binom{\dim X+1}{i} \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X+1-i)} \rangle \\ & \leq \sum_{i=0}^{\dim X+1} \binom{\dim X+1}{i} \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1)^{\cdot i} \cdot (\overline{\mathcal{D}}_2^{\text{ad}}; \mathcal{V}_2)^{\cdot (\dim X+1-i)} \rangle + \varepsilon. \end{aligned}$$

By Proposition 3.9, there exist $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{M}}_1) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_1; \mathcal{V})$ and $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{M}}_2) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_2; \mathcal{V})$ such that

$$\begin{aligned} & \sum_{i=0}^{\dim X+1} \binom{\dim X+1}{i} \langle (\overline{\mathcal{D}}_1^{\text{ad}}; \mathcal{V}_1)^{\cdot i} \cdot (\overline{\mathcal{D}}_2^{\text{ad}}; \mathcal{V}_2)^{\cdot (\dim X+1-i)} \rangle \\ & \leq \sum_{i=0}^{\dim X+1} \binom{\dim X+1}{i} \widehat{\deg}(\overline{\mathcal{M}}_1^{\cdot i} \cdot \overline{\mathcal{M}}_2^{\cdot (\dim X+1-i)}) + \varepsilon \end{aligned}$$

$$\begin{aligned}
&= \widehat{\deg} \left((\overline{\mathcal{M}}_1 + \overline{\mathcal{M}}_2)^{\cdot(\dim X + 1)} \right) + \varepsilon \\
&\leq \widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2) + \varepsilon.
\end{aligned}$$

By the same arguments, one can also show the assertions (2), (3), and (4). \square

Let X be a normal projective K -variety and let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$. We define

$$(4.1) \quad s_i := \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot(\dim X + 1 - i)} \rangle$$

for $i = 0, \dots, \dim X + 1$,

$$\begin{aligned}
(4.2) \quad r &= r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) \\
&:= \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (\mu^* \overline{D}_1 - t \overline{M}; \mathcal{V}_1^\mu) \succeq 0 \}
\end{aligned}$$

(inradius), and

$$(4.3) \quad R = R((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) := \frac{1}{r((\overline{D}_2; \mathcal{V}_2), (\overline{D}_1; \mathcal{V}_1))}$$

(circumradius).

Lemma 4.6. *Let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$.*

(1) *Let $(\overline{D}'_1; \mathcal{V}'_1), (\overline{D}'_2; \mathcal{V}'_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$. If $(\overline{D}_1; \mathcal{V}_1) \preceq (\overline{D}'_1; \mathcal{V}'_1)$ and $(\overline{D}_2; \mathcal{V}_2) \succeq (\overline{D}'_2; \mathcal{V}'_2)$, then*

$$r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) \leq r((\overline{D}'_1; \mathcal{V}'_1), (\overline{D}'_2; \mathcal{V}'_2)).$$

(2) *For $a > 0$, one has*

$$r(a(\overline{D}_1; \mathcal{V}_1), a(\overline{D}_2; \mathcal{V}_2)) = ar((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)).$$

Proof. (1): Obviously, one has

$$r((\overline{D}'_1; \mathcal{V}'_1), (\overline{D}'_2; \mathcal{V}'_2)) \geq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)).$$

For any $(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}'_2; \mathcal{V}'_2)$, $(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)$, so

$$\begin{aligned}
r((\overline{D}_1; \mathcal{V}_1), (\overline{D}'_2; \mathcal{V}'_2)) &= \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}'_2; \mathcal{V}'_2)} \sup \{ t \in \mathbb{R} : (\mu^* \overline{D}_1 - t \overline{M}; \mathcal{V}_1^\mu) \succeq 0 \} \\
&\geq \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (\mu^* \overline{D}_1 - t \overline{M}; \mathcal{V}_1^\mu) \succeq 0 \} \\
&= r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)).
\end{aligned}$$

(2): Note that $(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)$ if and only if $(\mu, a\overline{M}) \in \widehat{\Theta}(a\overline{D}_2; a\mathcal{V}_2)$. So

$$\begin{aligned}
&ar((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) \\
&= a \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (\mu^* \overline{D}_1 - t \overline{M}; \mathcal{V}_1^\mu) \succeq 0 \} \\
&= a \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (a\mu^* \overline{D}_1 - at \overline{M}; a\mathcal{V}_1^\mu) \succeq 0 \} \\
&= \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(a\overline{D}_2; a\mathcal{V}_2)} \sup \{ t \in \mathbb{R} : (\mu^* (a\overline{D}_1) - t \overline{M}; a\mathcal{V}_1^\mu) \succeq 0 \} \\
&= r(a(\overline{D}_1; \mathcal{V}_1), a(\overline{D}_2; \mathcal{V}_2)).
\end{aligned}$$

\square

Lemma 4.7. *Let $(\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}') \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, and let $\overline{P} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$.*

(1) *One has*

$$r((\overline{D}; \mathcal{V}), \overline{P}) = \sup \{t \in \mathbb{R} : (\overline{D} - t\overline{P}; \mathcal{V}) \succeq 0\}.$$

(2) *One has*

$$r((\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}'), \overline{P}) \geq r((\overline{D}; \mathcal{V}), \overline{P}) + r((\overline{D}'; \mathcal{V}'), \overline{P}).$$

(3) *Let $\overline{E}_i \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, $\mathcal{W}_j \in \text{BC}_{\mathbb{R}}(X)$, $v_k \in M_K \cup \{\infty\}$, and $\varphi_k \in C_{v_k}^0(X)$. One has*

$$\begin{aligned} \lim_{\varepsilon_i, \delta_j, \|\varphi_k\|_{\text{sup}} \rightarrow 0} r\left(\left(\overline{D} + \sum_{i=1}^m \varepsilon_i \overline{E}_i + \sum_{k=1}^l (0, \varphi_k[v_k]); \mathcal{V} + \sum_{j=1}^n \delta_j \mathcal{W}_j\right), \overline{P}\right) \\ = r((\overline{D}; \mathcal{V}), \overline{P}). \end{aligned}$$

Proof. The assertion (1) is obvious.

(2): If $(\overline{D} - t\overline{P}; \mathcal{V}) \succeq 0$ and $(\overline{D}' - t'\overline{P}; \mathcal{V}') \succeq 0$, then

$$(\overline{D} + \overline{D}' - (t + t')\overline{P}; \mathcal{V} + \mathcal{V}') \succeq 0.$$

So $r((\overline{D} + \overline{D}'; \mathcal{V} + \mathcal{V}'), \overline{P}) \geq t + t'$.

(3): The function $\widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \mathbb{R}$,

$$(\overline{D}; \mathcal{V}) \mapsto r((\overline{D}; \mathcal{V}), \overline{P}),$$

is concave. So the assertion results from Theorem 2.21(1) and [12, Theorem 6.3.4]. \square

Lemma 4.8. *Let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$, let U be a non-empty open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{D}_2 exists, and let $\delta > 0$. One has*

$$\begin{aligned} r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) \\ = \inf_{(\mu, \overline{H}) \in \widehat{\Theta}_{\text{amp}}(\overline{D}_2; \mathcal{V}_2)} \sup \{t \in \mathbb{R} : (\mu^* \overline{D}_1 - t\overline{H}; \mathcal{V}_1^\mu) \succeq 0\} \\ = \inf_{(\mathcal{X}, \mathcal{D}_2) \in \widehat{\Theta}_{U, \delta}(\overline{D}_2)} \inf_{(\tilde{\mu}, \overline{\mathcal{H}}) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_2; \mathcal{V})} \sup \{t \in \mathbb{R} : (\mu^* \overline{D}_1 - t\overline{\mathcal{H}}^{\text{ad}}; \mathcal{V}_1^\mu) \succeq 0\}. \end{aligned}$$

Proof. First, we show the first equality. The inequality \leq is clear. By definition, given any $\varepsilon > 0$, there exists a $(\mu : X' \rightarrow X, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)$ such that

$$(4.4) \quad r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{M}) \leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + \varepsilon.$$

By Lemma 4.7, there exists a sufficiently small $\delta > 0$ such that

$$(4.5) \quad r((1 + \delta)(\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{M}) \leq r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{M}) + \varepsilon.$$

By the same arguments as in Proposition 3.9(3), there exists an ample adelic \mathbb{R} -Cartier divisor \overline{H} on X' such that $(\mu, \overline{H}) \in \widehat{\Theta}_{\text{amp}}(\overline{D}_2; \mathcal{V}_2)$ and $(1 + \delta)\overline{H} \succeq \overline{M}$, so, by Lemma 4.6(1), (2),

$$\begin{aligned} (4.6) \quad r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}) &\leq (1 + \delta)r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}) \\ &= r((1 + \delta)(\mu^* \overline{D}_1; \mathcal{V}_1^\mu), (1 + \delta)\overline{H}) \\ &\leq r((1 + \delta)(\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{M}). \end{aligned}$$

All in all, one has

$$\begin{aligned} \inf_{(\mu, \overline{M}) \in \widehat{\Theta}_{\text{amp}}(\overline{D}_2; \mathcal{V}_2)} r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{M}) &\leq r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}) \\ &\leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + 2\varepsilon \end{aligned}$$

for every $\varepsilon > 0$.

Next, we show the second equality. The inequality \leq is clear. By the assertion (1), given any $\varepsilon > 0$, there exists a $(\mu : X' \rightarrow X, \overline{H}) \in \widehat{\Theta}_{\text{amp}}(\overline{D}_2; \mathcal{V}_2)$ such that $(\mu^* \overline{D}_2 - \overline{H}; \mathcal{V}_2^\mu)$ is big and

$$(4.7) \quad r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}) \leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + \varepsilon.$$

By Lemma 4.7(2), we can choose a $\delta_1 > 0$ such that

$$(4.8) \quad r((1 + \delta_1)(\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}) \leq r((\overline{D}_1; \mathcal{V}_1), \overline{H}) + \varepsilon.$$

Let U be a non-empty open subset of $\text{Spec}(O_K)$ over which models of definition for both \overline{D}_2 and \overline{H} exist. By Theorem 2.21(1), there exists a sufficiently small δ_2 such that $0 < \delta_2 \leq \delta$ and

$$\frac{\delta_1}{1 + \delta_1} \overline{H} - \delta_2 \sum_{v \in M_K \setminus U} (0, [v])$$

is big. By Proposition 3.9(4), there exist $(\mathcal{X}, \mathcal{D}_2) \in \widehat{\Theta}_{U, \delta_2}(\overline{D}_2)$ and $(\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{H}}) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_2; \mathcal{V}_2)$ such that

$$\frac{1}{1 + \delta_1} \overline{H} \preceq \overline{H} - \delta_2 \sum_{v \in M_K \setminus U} (0, [v]) \preceq \overline{\mathcal{H}}^{\text{ad}} \preceq \overline{H},$$

so

$$(4.9) \quad \begin{aligned} r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{\mathcal{H}}^{\text{ad}}) &\leq (1 + \delta_1) r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{\mathcal{H}}^{\text{ad}}) \\ &\leq r((1 + \delta_1)(\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}). \end{aligned}$$

All in all, we have

$$\begin{aligned} \inf_{(\mathcal{X}, \mathcal{D}_2) \in \widehat{\Theta}_{U, \delta}(\overline{D}_2)} \inf_{(\tilde{\mu}, \overline{\mathcal{H}}) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_2; \mathcal{V}_2)} r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{H}^{\text{ad}}) \\ \leq r((\mu^* \overline{D}_1; \mathcal{V}_1^\mu), \overline{\mathcal{H}}^{\text{ad}}) \\ \leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + 2\varepsilon \end{aligned}$$

for every $\varepsilon > 0$. □

The following gives a generalization of [18, Theorem 7.1 and Corollary 7.3] (see also [5], [10, Theorems 6.9 and 6.10], [26]).

Theorem 4.9. *We keep the notations as above. Suppose that $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{D}}\widehat{\text{Big}}_{\mathbb{R}, \mathbb{R}}(X)$.*

(1) *(An arithmetic Diskant inequality) One has*

$$0 \leq \left(s_{\frac{\dim X}{\dim X}}^{\frac{1}{\dim X}} - r s_0^{\frac{1}{\dim X}} \right)^{\dim X + 1} \leq s_{\dim X}^{1 + \frac{1}{\dim X}} - s_{\dim X + 1} \cdot s_0^{\frac{1}{\dim X}}.$$

(2) One has

$$\begin{aligned} \frac{s_{\dim X}^{\frac{1}{\dim X}} - \left(s_{\dim X}^{1+\frac{1}{\dim X}} - s_{\dim X+1} \cdot s_0^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}{s_0^{\frac{1}{\dim X}}} &\leq r \\ &\leq \frac{s_{\dim X+1}}{s_{\dim X}} \leq \dots \leq \frac{s_1}{s_0} \\ &\leq R \leq \frac{s_{\dim X+1}^{\frac{1}{\dim X}}}{s_1^{\frac{1}{\dim X}} - \left(s_1^{1+\frac{1}{\dim X}} - s_0 \cdot s_{\dim X+1}^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}} \end{aligned}$$

(3) (An arithmetic Bonnesen inequality) If $\dim X = 1$, then

$$\frac{s_0^2}{4}(R-r)^2 \leq s_1^2 - s_0 s_2.$$

Proof. (1): We divide the proof into three steps.

Step 1. In this step, we assume that $(\overline{D}_2; \mathcal{V}_2)$ is given by $\overline{P} \in \widehat{\text{Nef}}_{\mathbb{R}}(X) \cap \widehat{\text{Big}}_{\mathbb{R}}(X)$. By the global continuity of $\widehat{\text{vol}} : \widehat{\text{DDiv}}_{\mathbb{R}, \mathbb{R}}(X) \rightarrow \mathbb{R}$ (see Remark 2.26 and [20]), one has

$$\widehat{\text{vol}}(\overline{D}_1 - t\overline{P}; \mathcal{V}_1) \begin{cases} > 0 & \text{if } t < s, \\ = 0 & \text{if } t = s. \end{cases}$$

So, by Theorem 4.2,

$$(4.10) \quad \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1) = (\dim X + 1) \int_{t=0}^s \langle (\overline{D}_1 - t\overline{P}; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} dt.$$

On the other hand, for $t < s$, one has

$$(4.11) \quad \begin{aligned} 0 &\leq \langle (\overline{D}_1 - t\overline{P}; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} \\ &\leq \left(\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} \right)^{\frac{1}{\dim X}} - t \widehat{\text{vol}}(\overline{P})^{\frac{1}{\dim X}} \Big)^{\dim X} \end{aligned}$$

by Corollary 4.4(2). All in all, one has

$$\begin{aligned} &\widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1) \widehat{\text{vol}}(\overline{P})^{\frac{1}{\dim X}} \\ &\leq (\dim X + 1) \widehat{\text{vol}}(\overline{P})^{\frac{1}{\dim X}} \\ &\quad \times \int_{t=0}^s \left(\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} \right)^{\frac{1}{\dim X}} - t \widehat{\text{vol}}(\overline{P})^{\frac{1}{\dim X}} \Big)^{\dim X} dt \\ &= \left(\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} \right)^{1+\frac{1}{\dim X}} \\ &\quad - \left(\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot \dim X} \rangle \cdot \overline{P} \right)^{\frac{1}{\dim X}} - s \widehat{\text{vol}}(\overline{P})^{\frac{1}{\dim X}} \Big)^{\dim X+1}. \end{aligned}$$

Step 2. In this step, we show the following claim.

Claim 4.10. *Let $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BBig}}_{\mathbb{R}, \mathbb{R}}(X)$. For any $i, j \geq 0$ with $i + j \leq \dim X + 1$, one has*

$$r^j s_i \leq s_{i+j}.$$

Proof of Claim 4.10. We can assume $i > 0$. Given any ε with $0 < \varepsilon < s_i/2$, there exists a $(\mu : X' \rightarrow X, \overline{M}) \in \widehat{\Theta}(\overline{D}_2; \mathcal{V}_2)$ such that

$$(4.12) \quad \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot \overline{M}^j \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X+1-(i+j))} \rangle + \varepsilon \geq s_i.$$

Set $r' := r((\overline{D}_1; \mathcal{V}_1), \overline{M}) \geq r$. Since $(\mu^* \overline{D}_1 - r' \overline{M}; \mathcal{V}_1^\mu) \succeq 0$, one has, by Remark 3.10(2),(3),

$$(4.13) \quad \begin{aligned} r'^j \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot \overline{M}^{\cdot j} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X + 1 - (i+j))} \rangle \\ = \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (r' \overline{M})^{\cdot j} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X + 1 - (i+j))} \rangle \\ \leq s_{i+j} \end{aligned}$$

Hence,

$$\begin{aligned} r^j s_i &\leq r'^j \cdot \left(\langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot \overline{M}^{\cdot j} \cdot (\overline{D}_2; \mathcal{V}_2)^{\cdot (\dim X + 1 - (i+j))} \rangle + \varepsilon \right) \\ &\leq s_{i+j} \left(1 + \frac{2\varepsilon}{s_i} \right) \end{aligned}$$

for every sufficiently small $\varepsilon > 0$. \square

Step 3. In general, we take an arbitrary $\varepsilon > 0$. Let U be a non-empty open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{D}_2 exists. By continuity (see Proposition 3.11), there exists a $\delta > 0$ such that, for every $(\mathcal{X}, \mathcal{D}) \in \widehat{\Theta}_{U, \delta}(\overline{D}_2)$,

$$(4.14) \quad \begin{aligned} &\frac{s_{\dim X}^{\frac{1}{\dim X}} - \left(s_{\dim X}^{1 + \frac{1}{\dim X}} - s_{\dim X + 1} \cdot s_0^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X + 1}}}{s_0^{\frac{1}{\dim X}}} \\ &\leq \frac{s'_{\dim X}{}^{\frac{1}{\dim X}} - \left(s'_{\dim X}{}^{1 + \frac{1}{\dim X}} - s'_{\dim X + 1} \cdot s_0'^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X + 1}}}{s_0'^{\frac{1}{\dim X}}} + \varepsilon, \end{aligned}$$

where $s'_i := \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (\overline{\mathcal{D}}^{\text{ad}}; \mathcal{V}_2)^{\cdot (\dim X + 1 - i)} \rangle$.

By Lemma 4.8, there exist a $(\mathcal{X}, \mathcal{D}_0) \in \widehat{\Theta}_{U, \delta}(\overline{D}_2)$ and a $(\tilde{\varphi} : \mathcal{X}' \rightarrow \mathcal{X}, \overline{\mathcal{M}}_0) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_0; \mathcal{V})$ such that

$$(4.15) \quad r(\tilde{\varphi}_*^{-1}(\overline{D}_1; \mathcal{V}_1), \overline{\mathcal{M}}_0^{\text{ad}}) \leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + \varepsilon.$$

Moreover, by Proposition 3.9(2), there exists a $(\tilde{\mu} : \mathcal{X}'' \rightarrow \mathcal{X}, \overline{\mathcal{M}}) \in \widehat{\Theta}_{\text{ad}}(\overline{\mathcal{D}}_0; \mathcal{V})$ such that $\tilde{\mu}$ factorizes into $\mathcal{X}'' \xrightarrow{\psi} \mathcal{X}' \xrightarrow{\tilde{\varphi}} \mathcal{X}$, $\overline{\mathcal{M}} \geq \tilde{\psi}^* \overline{\mathcal{M}}_0$, and

$$(4.16) \quad \begin{aligned} &\frac{s'_{\dim X}{}^{\frac{1}{\dim X}} - \left(s'_{\dim X}{}^{1 + \frac{1}{\dim X}} - s'_{\dim X + 1} \cdot s_0'^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X + 1}}}{s_0'^{\frac{1}{\dim X}}} \\ &\leq \frac{s''_{\dim X}{}^{\frac{1}{\dim X}} - \left(s''_{\dim X}{}^{1 + \frac{1}{\dim X}} - s''_{\dim X + 1} \cdot s_0''^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X + 1}}}{s_0''^{\frac{1}{\dim X}}} + \varepsilon, \end{aligned}$$

where $s''_i := \langle (\overline{D}_1; \mathcal{V}_1)^{\cdot i} \cdot (\overline{\mathcal{M}}^{\text{ad}})^{\cdot (\dim X + 1 - i)} \rangle$.

By Step 1 and Lemma 4.6(1), one has

$$(4.17) \quad \frac{s''_{\dim X}{}^{\frac{1}{\dim X}} - \left(s''_{\dim X}{}^{1 + \frac{1}{\dim X}} - s''_{\dim X + 1} \cdot s_0''^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X + 1}}}{s_0''^{\frac{1}{\dim X}}}$$

$$\begin{aligned}
&\leq r(\tilde{\mu}_*^{-1}(\overline{D}_1; \mathcal{V}_1), \overline{\mathcal{M}}^{\text{ad}}) \\
&\leq r(\tilde{\varphi}_*^{-1}(\overline{D}_1; \mathcal{V}_1), \overline{\mathcal{M}}_0^{\text{ad}}).
\end{aligned}$$

All in all,

$$\frac{s_{\dim X}^{\frac{1}{\dim X}} - \left(s_{\dim X}^{1 + \frac{1}{\dim X}} - s_{\dim X+1} \cdot s_0^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}{s_0^{\frac{1}{\dim X}}} \leq r((\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2)) + 3\varepsilon$$

for every $\varepsilon > 0$.

(2): By the assertion (1) and Claim 4.10, one has

$$\frac{s_{\dim X}^{\frac{1}{\dim X}} - \left(s_{\dim X}^{1 + \frac{1}{\dim X}} - s_{\dim X+1} \cdot s_0^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}{s_0^{\frac{1}{\dim X}}} \leq r \leq \frac{s_{\dim X+1}}{s_{\dim X}}.$$

By applying the above inequalities to $r((\overline{D}_2; \mathcal{V}_2), (\overline{D}_1; \mathcal{V}_1)) = 1/R$, one has

$$\frac{s_1^{\frac{1}{\dim X}} - \left(s_1^{1 + \frac{1}{\dim X}} - s_0 \cdot s_{\dim X+1}^{\frac{1}{\dim X}} \right)^{\frac{1}{\dim X+1}}}{s_{\dim X+1}^{\frac{1}{\dim X}}} \leq \frac{1}{R} \leq \frac{s_0}{s_1}.$$

Moreover, by Proposition 4.5(2),

$$\frac{s_{\dim X+1}}{s_{\dim X}} \leq \frac{s_{\dim X}}{s_{\dim X-1}} \leq \dots \leq \frac{s_2}{s_1} \leq \frac{s_1}{s_0}.$$

So we conclude.

(3): By the assertion (2), one has

$$\begin{aligned}
\frac{s_0^2}{4}(R-r)^2 &\leq \frac{s_0^2}{4} \left(\frac{s_2}{s_1 - \sqrt{s_1^2 - s_0 \cdot s_2}} - \frac{s_1 - \sqrt{s_1^2 - s_0 \cdot s_2}}{s_0} \right) \\
&= s_1^2 - s_0 s_2.
\end{aligned}$$

□

Remark 4.11. (1) If $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{D}\text{Big}}_{\mathbb{R}, \mathbb{R}}(X)$ satisfies

$$\widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2)^{\frac{1}{\dim X+1}} = \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1)^{\frac{1}{\dim X+1}} + \widehat{\text{vol}}(\overline{D}_2; \mathcal{V}_2)^{\frac{1}{\dim X+1}},$$

then $s_{\dim X}^{\dim X+1} = s_{\dim X+1}^{\dim X} \cdot s_0$, $s_1^{\dim X+1} = s_0^{\dim X} \cdot s_{\dim X}$, and

$$\left(\frac{s_{\dim X}}{s_0} \right)^{\frac{1}{\dim X}} = r = \frac{s_{\dim X+1}}{s_{\dim X}} = \dots = \frac{s_1}{s_0} = R = \left(\frac{s_{\dim X+1}}{s_1} \right)^{\frac{1}{\dim X}},$$

but the converse may not be true.

(2) Suppose that X is a smooth curve. Every discrete valuation is divisorial, so that we can naturally identify the three types of base conditions

$$\text{Div}_{\mathbb{R}}(X) = \text{WDiv}_{\mathbb{R}}(X) = \text{BC}_{\mathbb{R}}(X).$$

For $(\overline{D}; \mathcal{V}) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, we put

$$\Upsilon(\overline{D}; \mathcal{V}) := \{ \overline{P} : \overline{P} \text{ is nef and } \overline{P} \leq (\overline{D}; \mathcal{V}) \}.$$

If $\Upsilon(\overline{D}; \mathcal{V}) \neq \emptyset$, then it is known that $\Upsilon(\overline{D}; \mathcal{V})$ admits a unique maximal element $\overline{P}(\overline{D}; \mathcal{V})$ (see [25, Theorem 6.2.3]).

For $(\overline{D}_1; \mathcal{V}_1), (\overline{D}_2; \mathcal{V}_2) \in \widehat{\text{BDiv}}_{\mathbb{R}, \mathbb{R}}(X)$, the following are equivalent.

- (a) $\widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2; \mathcal{V}_1 + \mathcal{V}_2)^{\frac{1}{2}} = \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1)^{\frac{1}{2}} + \widehat{\text{vol}}(\overline{D}_2; \mathcal{V}_2)^{\frac{1}{2}}.$
- (b) $\overline{P}(\overline{D}_1; \mathcal{V}_1) / \widehat{\text{vol}}(\overline{D}_1; \mathcal{V}_1)^{\frac{1}{2}} \sim_{\mathbb{R}} \overline{P}(\overline{D}_2; \mathcal{V}_2) / \widehat{\text{vol}}(\overline{D}_2; \mathcal{V}_2)^{\frac{1}{2}}.$

APPENDIX A. AN ARITHMETIC NAKAI–MOISHEZON CRITERION OVER CURVES

In this appendix, we show an arithmetic Nakai–Moishezon criterion for adelic \mathbb{R} -Cartier divisors on curves (Corollary A.4).

Lemma A.1. *Let \mathcal{X} be a regular and geometrically connected arithmetic surface over $\text{Spec}(O_K)$ with smooth generic fiber $X := \mathcal{X}_K$. Let D be an ample \mathbb{R} -Cartier divisor on X . There exist a $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ and a relatively ample \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D}|_X = D + (\phi)$.*

Proof. Given any closed point x' in a closed fiber \mathcal{X}_P over $P \in \text{Spec}(O_K)$, one can find an $x \in X$ such that the Zariski closure $\overline{\{x\}}$ in \mathcal{X} contains x' . In fact, let $\varpi \in \mathcal{O}_{\mathcal{X}, x'}$ be a local equation defining \mathcal{X}_P around x' , and choose an $f \in \mathcal{O}_{\mathcal{X}, x'}$ such that ϖ, f form a system of parameters for $\mathcal{O}_{\mathcal{X}, x'}$. Then $f\mathcal{O}_{\mathcal{X}, x'}$ is a prime ideal of height one, and does not contain ϖ . (See also [16, Theorem 4.1].)

There exist finitely many $P_1, \dots, P_l \in \text{Spec}(O_K)$ such that \mathcal{X} is geometrically irreducible over $\text{Spec}(O_K) \setminus \{P_1, \dots, P_l\}$. By the above argument, there exist $x_1, \dots, x_m \in X$ such that $\bigcup \overline{\{x_i\}}$ meets every irreducible component of $\mathcal{X}_{P_1}, \dots, \mathcal{X}_{P_l}$. We take a $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that $D + (\phi)$ can be written as a sum $\sum_{j=1}^n a_j D_j$ such that $n \geq 1$, $a_j > 0$, D_j are prime Cartier divisors on X , and $\bigcup \text{Supp}(D_j) \supset \{x_1, \dots, x_m\}$. Let \mathcal{D}_j be the Zariski closure of D_j in \mathcal{X} . Then

$$\mathcal{D} := \sum_{j=1}^n a_j \mathcal{D}_j$$

is a relatively ample \mathbb{R} -Cartier divisor on \mathcal{X} extending $D + (\phi)$. \square

Lemma A.2. *Let X be a smooth and geometrically irreducible K -curve and let $\overline{D} = \left(D, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}}[v]\right)$ be an adelic \mathbb{R} -Cartier divisor on X such that D is ample. The following are equivalent.*

- (1) *For every $\varepsilon > 0$, there exists an O_K -model $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ of (X, D) such that $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ extends a fixed model of definition for \overline{D} , \mathcal{D}_ε is a relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}_ε , and*

$$\|g_v^{\overline{D}} - g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)}\|_{\text{sup}} \leq \varepsilon$$

for every $v \in M_K$.

- (2) *For every $\varepsilon > 0$, there exists an O_K -model $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ of (X, D) such that $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ extends a fixed model of definition for \overline{D} , \mathcal{D}_ε is a relatively ample \mathbb{R} -Cartier divisor on \mathcal{X}_ε , and*

$$\|g_v^{\overline{D}} - g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)}\|_{\text{sup}} \leq \varepsilon$$

for every $v \in M_K$.

Proof. The implication (2) \Rightarrow (1) is obvious, so that we are going to show the converse. There exist a non-empty open subset U of $\text{Spec}(O_K)$ and a U -model of definition $(\mathcal{X}_U, \mathcal{D}_U)$ for \overline{D} such that $\mathcal{X}_U \rightarrow U$ is smooth.

By the condition (1), for any $\varepsilon > 0$, there exists an O_K -model $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ such that $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ extends $(\mathcal{X}_U, \mathcal{D}_U)$, \mathcal{D}_ε is a relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}_ε , and

$$\|g_v^{\overline{D}} - g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)}\|_{\sup} \leq \frac{\varepsilon}{2}$$

for every $v \in M_K$.

By desingularization [21, page 55], there exists a birational morphism $\pi_\varepsilon : \mathcal{X}'_\varepsilon \rightarrow \mathcal{X}_\varepsilon$ such that \mathcal{X}'_ε is regular and π_ε is isomorphic over U . By Lemma A.1, there exists a relatively ample \mathbb{R} -Cartier divisor \mathcal{D}'_ε on \mathcal{X}'_ε that extends D . We can choose a sufficiently small $\delta_\varepsilon > 0$ such that

$$\left\| g_v^{\overline{D}} - \left((1 - \delta_\varepsilon) g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)} + \delta_\varepsilon g_v^{(\mathcal{X}'_\varepsilon, \mathcal{D}'_\varepsilon)} \right) \right\|_{\sup} \leq \varepsilon$$

for every $v \in M_K$. So $(\mathcal{X}'_\varepsilon, (1 - \delta_\varepsilon)\pi_\varepsilon^* \mathcal{D}_\varepsilon + \delta_\varepsilon \mathcal{D}'_\varepsilon)$ is an O_K -model of (X, D) having the required properties. \square

The following is an arithmetic analogue of the theorem of Campana–Petrernell [8, Theorem 1.3].

Theorem A.3. *Let X be a smooth projective K -variety and let $\overline{D} = \left(D, \sum_{v \in M_K \cup \{\infty\}} g_v^{\overline{D}}[v] \right)$ be a adelic \mathbb{R} -Cartier divisor on X such that the following condition $(*)$ is satisfied.*

- (*) For every $\varepsilon > 0$, there exists an O_K -model $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ of (X, D) such that $(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)$ extends a fixed model of definition for \overline{D} , \mathcal{D}_ε is a relatively ample \mathbb{R} -Cartier divisor on \mathcal{X}_ε , and*

$$\|g_v^{\overline{D}} - g_v^{(\mathcal{X}_\varepsilon, \mathcal{D}_\varepsilon)}\|_{\sup} \leq \varepsilon$$

for every $v \in M_K$.

Then the following are equivalent.

- (1) \overline{D} is w -ample.
- (2) \overline{D} is ample

Proof. (1) \Rightarrow (2) is nothing but Theorem 2.11(1), so we are going to show the converse.

(2) \Rightarrow (1): If $\dim X = 0$, then \overline{D} can be written as $\left(0, \sum_{v \in M_K \cup \{\infty\}} \lambda_v[v] \right)$, where $\lambda_v = 0$ for all but finitely many $v \in M_K$. By the arithmetic Riemann-Roch formula, there exists a $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\overline{D} + \widehat{(\phi)} > 0$.

We show the theorem by induction on dimension. We can assume that \overline{D} is associated to an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ on a normal O_K -model \mathcal{X} such that \mathcal{D} is relatively ample, such that $g_\infty^{\overline{\mathcal{D}}}$ is of C^∞ -type and positive pointwise, and such that

$$(A.1) \quad \inf_{x \in X(\overline{K})} h_{\overline{\mathcal{D}}}(x) > 0.$$

In fact, by Theorem 2.11(3), we have $\lambda := \inf_{x \in X(\overline{K})} h_{\overline{D}}(x) > 0$. Let U be an open subset of $\text{Spec}(O_K)$ over which the fixed model of definition $(\mathcal{X}_U, \mathcal{D}_U)$ of \overline{D} exists. Let ε be a positive real number such that

$$\varepsilon (\#(\text{Spec}(O_K) \setminus U) + [K : \mathbb{Q}]) < \lambda.$$

By the condition $(*)$ and the regularization theorem (see [24, Theorem 4.6]), there exist a normal O_K -model \mathcal{X}_ε and an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}_\varepsilon$ on \mathcal{X}_ε such

that \mathcal{D}_ε is relatively ample, such that $g_{\infty^\varepsilon}^{\overline{\mathcal{D}}}$ is of C^∞ -type and positive pointwise, and such that

$$\overline{\mathcal{D}}_\varepsilon^{\text{ad}} \leq \overline{D} \leq \overline{\mathcal{D}}_\varepsilon^{\text{ad}} + 2\varepsilon \sum_{v \in (M_K \setminus U) \cup \{\infty\}} (0, [v]).$$

Note that

$$\inf_{x \in X(\overline{K})} h_{\overline{\mathcal{D}}_\varepsilon}(x) \geq \lambda - (\sharp(\text{Spec}(O_K) \setminus U) + [K : \mathbb{Q}])\varepsilon > 0,$$

and that, if $\overline{\mathcal{D}}_\varepsilon^{\text{ad}}$ is w-ample, then so is \overline{D} .

By induction hypothesis, $\overline{\mathcal{D}}|_{\mathcal{Y}}$ is w-ample for every horizontal arithmetic subvariety \mathcal{Y} with $\dim \mathcal{Y} < \dim \mathcal{X}$. Let $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$ be nef and w-ample arithmetic Cartier divisors of C^∞ -type on \mathcal{X} such that $\overline{\mathcal{D}}$ is contained in the rational \mathbb{R} -subspace of $\text{Div}_{\mathbb{R}}(\mathcal{X})$ spanned by $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$. There exist positive real numbers $\varepsilon_1, \dots, \varepsilon_l$ such that

$$(A.2) \quad \overline{\mathcal{E}} := \overline{\mathcal{D}} - \sum_{i=1}^l \varepsilon_i \overline{\mathcal{A}}_i$$

is rational, $\overline{\mathcal{E}}$ is ample, $g_{\infty}^{\overline{\mathcal{E}}}$ is positive pointwise, and

$$(\dim X + 1) \cdot \sum_{i=1}^l \varepsilon_i \widehat{\deg} \left(\overline{\mathcal{D}}^{\dim X} \cdot \overline{\mathcal{A}}_i \right) < \widehat{\text{vol}}(\overline{\mathcal{D}}).$$

Set $\overline{\mathcal{A}} := \sum_{i=1}^l \varepsilon_i \overline{\mathcal{A}}_i$. By the arithmetic Siu inequality (Proposition 4.1(1)), we have

$$\widehat{\text{vol}}(\overline{\mathcal{E}}) \geq \widehat{\text{vol}}(\overline{\mathcal{D}}) - (\dim X + 1) \cdot \widehat{\deg} \left(\overline{\mathcal{D}}^{\dim X} \cdot \overline{\mathcal{A}} \right) > 0,$$

so that there exists an $s \in \widehat{\Gamma}_{\mathbb{Q}}^{\text{ss}}(\overline{\mathcal{E}}^{\text{ad}}) \setminus \{0\}$.

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ be the reduced, irreducible, and horizontal components of $\text{Supp}_{\mathbb{C}}(\overline{\mathcal{E}} + (s))$. Since $\overline{\mathcal{D}}|_{\mathcal{Y}_1}, \dots, \overline{\mathcal{D}}|_{\mathcal{Y}_r}$ are all w-ample, one finds a sufficiently small $\delta > 0$ such that $\overline{\mathcal{D}} - \delta \overline{\mathcal{A}}$ is relatively ample and

$$\overline{\mathcal{D}}|_{\mathcal{Y}_1} - \delta \overline{\mathcal{A}}|_{\mathcal{Y}_1}, \dots, \overline{\mathcal{D}}|_{\mathcal{Y}_r} - \delta \overline{\mathcal{A}}|_{\mathcal{Y}_r}$$

are all w-ample (see Lemma 2.10(4)).

Set

$$(A.3) \quad \overline{\mathcal{F}} := \overline{\mathcal{D}} - \delta \overline{\mathcal{A}} = \overline{\mathcal{E}} + (1 - \delta) \overline{\mathcal{A}}.$$

Let $x \in X(\overline{K})$. If $x \notin \bigcup_{j=1}^p \mathcal{Y}_j(\overline{K})$, then $h_{\overline{\mathcal{F}}}(x) \geq h_{\overline{\mathcal{E}}}(x) > 0$. If $x \in \mathcal{Y}_j(\overline{K})$ for a j , then $h_{\overline{\mathcal{F}}}(x) > 0$ since $\overline{\mathcal{F}}|_{\mathcal{Y}_j}$ is w-ample. So $\overline{\mathcal{F}}$ is nef and $\overline{\mathcal{F}}$ is contained in the rational \mathbb{R} -subspace spanned by $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$. Hence, by Theorem 2.11(5), we conclude that

$$\overline{\mathcal{D}}^{\text{ad}} = \overline{\mathcal{F}}^{\text{ad}} + \delta \sum_{i=1}^l \varepsilon_i \overline{\mathcal{A}}_i^{\text{ad}}$$

is w-ample. □

As a consequence of Theorem A.3 and Lemma A.2, we have the following.

Corollary A.4. *Let X be a smooth K -curve and let \overline{D} be an adelic \mathbb{R} -Cartier divisor on X . The following are equivalent.*

- (1) \overline{D} is ample.

(2) \overline{D} is w -ample and relatively nef.

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